



## GRADE 8 MATH: EXPRESSIONS & EQUATIONS

### UNIT OVERVIEW

This unit builds directly from prior work on proportional reasoning in 6th and 7th grades, and extends the ideas more formally into the realm of algebra.

### TASK DETAILS

Task Name: Expressions and Equations

Grade: 8

Subject: Mathematics

Task Description: This sequence of tasks ask students to demonstrate and effectively communicate their mathematical understanding of ratios and proportional relationships, with a focus on expressions and equations. Their strategies and executions should meet the content, thinking processes and qualitative demands of the tasks.

#### Standards:

7.RP.2 Recognize and represent proportional relationships between quantities.

7.RP.2a Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

8.EE.5 Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

8.EE.6 Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y=mx$  for a line through the origin and the equation  $y=mx+b$  for a line intercepting the vertical axis at  $b$ .

8.EE.8c Solve real-world and mathematical problems leading to two linear equations in two variables.

8.F.2 Compare properties of two functions each represented in a different way.

8.F.4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

#### Standards for Mathematical Practice:

MP.1 Make sense of problems and persevere in solving them.

MP.2 Reason abstractly and quantitatively.

MP.3 Construct viable arguments and critique the reasoning of others.

MP.4 Model with mathematics.

MP.6 Attend to precision.



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The task and instructional supports in the following pages are designed to help educators understand and implement tasks that are embedded in Common Core-aligned curricula. While the focus for the 2011-2012 Instructional Expectations is on engaging students in Common Core-aligned culminating tasks, it is imperative that the tasks are embedded in units of study that are also aligned to the new standards. Rather than asking teachers to introduce a task into the semester without context, this work is intended to encourage analysis of student and teacher work to understand what alignment looks like. We have learned through this year's Common Core pilots that beginning with rigorous assessments drives significant shifts in curriculum and pedagogy. Universal Design for Learning (UDL) support is included to ensure multiple entry points for all learners, including students with disabilities and English language learners.

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Acknowledgements: The unit outline was developed by Kara Imm and Courtney Allison-Horowitz with input from Curriculum Designers Alignment Review Team. The tasks were developed by the 2010-2011 NYC DOE Middle School Performance Based Assessment Pilot Design Studio Writers, in collaboration with the Institute for Learning.



## GRADE 8 MATH: EXPRESSIONS & EQUATIONS PERFORMANCE TASK

NYC Grade 8 Assessment 1

Performance Based Assessment  
Equations and Expressions – Grade 8

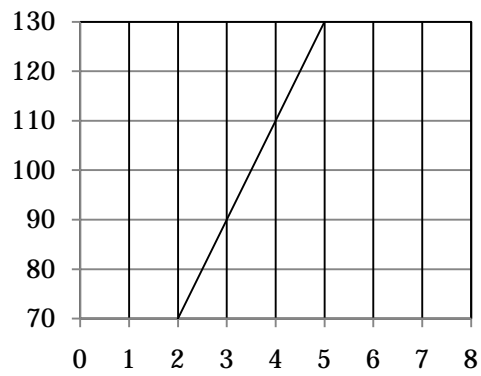
Name \_\_\_\_\_

School \_\_\_\_\_

Date \_\_\_\_\_

Teacher \_\_\_\_\_

1. Does the graph below represent a proportional relationship? Justify your response.



2. Kanye West expects to sell 350,000 albums in one week.
- How many albums will he have to sell every day in order to meet that expectation?
  - West has a personal goal of selling 5 million albums. If he continues to sell albums at the same rate, how long will it take him to achieve that goal? Explain how you made your decision.
  - The equation  $y = 40,000x$ , where  $x$  is the number of days and  $y$  is the number of albums sold, describes the number of albums another singer expects to sell. Does this singer expect to sell more or fewer albums than West? Justify your response.

3. Marvin likes to run from his home to the recording studio. He uses his iPod to track the time and distance he travels during his run. The table below shows the data he recorded during yesterday's run.

Time (Minute)	Distance
5	0.833
10	1.660
15	2.545
20	3.332
25	4.003
30	5.012
35	5.831

- Write an algebraic equation to model the data Marvin collected. Explain in words the reasoning you used to choose your equation.
- Does the data represent a proportional relationship? Explain your reasoning in words.
- If Marvin continues running at the pace indicated in your equation, how long will it take him to reach the recording studio, which is 12 miles from his home? Use mathematical reasoning to justify your response.

4. Jumel and Ashley have two of the most popular phones on the market, a Droid and an iPhone. Jumel's monthly cell phone plan is shown below, where  $c$  stands for the cost in dollars, and  $t$  stands for the number of texts sent each month.

$$\text{Jumel: } c = 60 + 0.05t$$

Ashley's plan costs \$.35 per text, in addition to a monthly fee of \$45.

- a. Whose plan, Jumel's or Ashley's, costs less if each of them sends 30 texts in a month? Explain how you determined your answer.
- b. How much will Ashley's plan cost for the same number of texts as when Jumel's costs \$75.00?
- c. Explain in writing how you know if there is a number of texts for which both plans cost the same amount.



# GRADE 8 MATH: EXPRESSIONS & EQUATIONS

## UNIVERSAL DESIGN FOR LEARNING (UDL)

### PRINCIPLES

**Math Grade 8 - Proportional Relationships, Lines, and Linear Equations**  
**Common Core Learning Standards/  
Universal Design for Learning**

The goal of using Common Core Learning Standards (CCLS) is to provide the highest academic standards to all of our students. Universal Design for Learning (UDL) is a set of principles that provides teachers with a structure to develop their instruction to meet the needs of a diversity of learners. UDL is a research-based framework that suggests each student learns in a unique manner. A one-size-fits-all approach is not effective to meet the diverse range of learners in our schools. By creating options for how instruction is presented, how students express their ideas, and how teachers can engage students in their learning, instruction can be customized and adjusted to meet individual student needs. In this manner, we can support our students to succeed in the CCLS.

Below are some ideas of how this Common Core Task is aligned with the three principles of UDL; providing options in representation, action/expression, and engagement. As UDL calls for multiple options, the possible list is endless. Please use this as a starting point. Think about your own group of students and assess whether these are options you can use.

**REPRESENTATION:** *The “what” of learning.* How does the task present information and content in different ways? How students gather facts and categorize what they see, hear, and read. How are they identifying letters, words, or an author's style?

*In this task, teachers can...*

- ✓ **Present key concepts in one form of symbolic representation (e.g., an expository text or a math equation) with an alternative form (e.g., an illustration, dance/movement, diagram, table, model, video, comic strip, storyboard, photograph, animation, physical or virtual manipulative) by** building a word wall or glossary of terms which include *interactive* examples and *online resources*.

**ACTION/EXPRESSION:** *The “how” of learning.* How does the task differentiate the ways that students can express what they know? How do they plan and perform tasks? How do students organize and express their ideas?

*In this task, teachers can...*

- ✓ **Embed coaches or mentors that model think-alouds of the process** by engaging students in paired learning, retelling, and modeling of solution-based discussions and questioning.

**Math Grade 8 - *Proportional Relationships, Lines, and Linear Equations***  
**Common Core Learning Standards/  
Universal Design for Learning**

**ENGAGEMENT:** *The “why” of learning.* How does the task stimulate interest and motivation for learning? How do students get engaged? How are they challenged, excited, or interested?

*In this task, teachers can...*

- ✓ **Vary activities and sources of information so that they can be personalized and contextualized to learners’ lives** by engaging students in discussions related to their own personal interests and relevant topics.

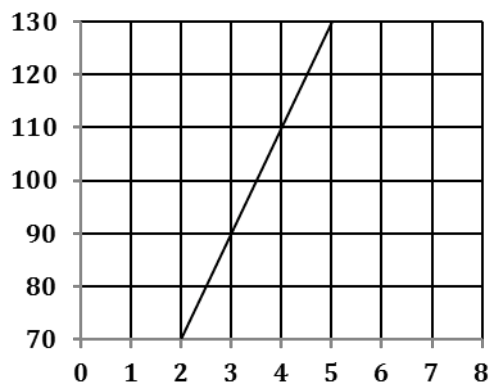
Visit <http://schools.nyc.gov/Academics/CommonCoreLibrary/default.htm> to learn more information about UDL.



## GRADE 8 MATH: EXPRESSIONS & EQUATIONS BENCHMARK PAPERS WITH RUBRICS

This section contains benchmark papers that include student work samples for each of the four tasks in the Expressions & Equations assessment. Each paper has descriptions of the traits and reasoning for the given score point, including references to the Mathematical Practices.

1. Does the graph below represent a proportional relationship? Use mathematical reasoning to justify your response.



NYC Grade 8 Assessment 1  
Determining Proportionality Task  
Benchmark Papers

3 Points

The response accomplishes the prompted purpose and effectively communicates the student's mathematical understanding. The student's strategy and execution meet the content (including concepts, technique, representations, and connections), thinking processes, and qualitative demands of the task. Minor omissions may exist, but do not detract from the correctness of the response.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them, and (2) Reason abstractly and quantitatively, since students need to abstract information from the graph, create a mathematical representation of the problem numerically or graphically, and consider whether the relationship is proportional. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem with tables, ratios and/or equations. Evidence of the Mathematical Practice, (6) Attend to precision, can include use of rise/run to extend the line and proper use of ratios. Evidence of the Mathematical Practice, (7) Look for and make use of structure, can include use of intentional techniques (rise/run) to extend the line and/or recognition that the equation of a proportional relationship is linear, with intercept equal to 0.

Either verbally or symbolically, the strategy used to solve the problem is stated, as is the work used to find ratios, intercept, or equation. Minor arithmetic errors may be present, but no errors of reasoning appear.

Justification may include reasoning as follows:

- a. Values from the clearly readable points on the graph are used to form ratios; the fact that the ratios are not equivalent is used to justify that the relationship is not proportional.

Readable Points				
x	2	3	4	5
y	70	90	110	130

- b. The graph is carefully extended to the y-intercept, possibly using rise/run. The fact that the graph does not pass through (0, 0) is used to justify that the relationship is not proportional.
- c. Techniques are used to determine the equation of the line,  $y = 20x + 30$ . The fact that the y-intercept is not 0 or the graph does not pass through (0, 0) is used to justify that the relationship is not proportional.

No, because they are not in proportion

$$\frac{3}{90} \neq \frac{4}{110}$$

$$\frac{4}{110} \neq \frac{5}{130}$$

$$\frac{3}{90} \neq \frac{5}{130}$$

NYC Grade 8 Assessment 1  
Determining Proportionality Task  
Benchmark Papers

2 Points

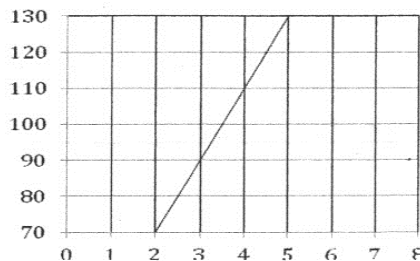
The response demonstrates adequate evidence of the learning and strategic tools necessary to complete the prompted purpose. It may contain overlooked issues, misleading assumptions, and/or errors in execution. Evidence in the response demonstrates that the student can revise the work to accomplish the task with the help of written feedback or dialogue.

Either verbally or symbolically, the strategy used to solve the problem is stated, as is the work used to find ratios, intercept, or equation. Reasoning may contain incomplete, ambiguous or misrepresentative ideas. Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them, and (2) Reason abstractly and quantitatively (since students need to abstract information from the graph, create a mathematical representation of the problem numerically or graphically, and consider whether the relationship is proportional.) Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem with tables, ratios and/or equations. Evidence of the Mathematical Practice, (6) Attend to precision, can include use of rise/run to extend the line and proper use of ratios. Evidence of the Mathematical Practice, (7) Look for and make use of structure, can include use of intentional techniques (rise/run) to extend the line and/or recognition that the equation of a proportional relationship is linear, with intercept equal to 0.

Justification may include reasoning as follows:

- Values from the clearly readable points on the graph are used to form ratios; however, it is not clear that the student is attempting to show the ratios are not equivalent.
- The graph is extended to the y-intercept, possibly using rise/run, to determine whether or not the graph passes through (0, 0), but errors in extension make the graph appear to pass through (0, 0).
- Techniques are incorrectly used to determine the equation of the line, but the value of the resulting y-intercept is then correctly used to justify that the relationship is or is not proportional.

Readable Points				
x	2	3	4	5
y	70	90	110	130



$$\frac{2}{70} \rightarrow \frac{3}{90}$$

$$210 \quad 180$$

No because if you take any two coordinates they're they are not equal

NYC Grade 8 Assessment 1  
Determining Proportionality Task  
Benchmark Papers

1 Point

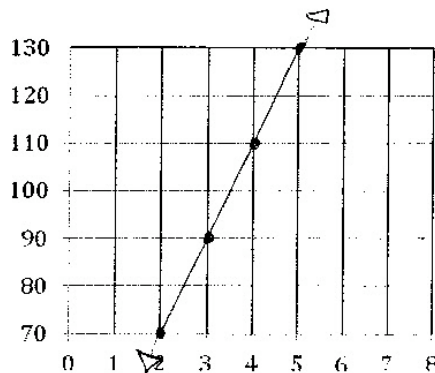
The response demonstrates some evidence of mathematical knowledge that is appropriate to the intent of the prompted purpose. An effort was made to accomplish the task, but with little success. Evidence in the response demonstrates that, with instruction, the student can revise the work to accomplish the task.

Some evidence of reasoning is demonstrated either verbally or symbolically, is often based on misleading assumptions, and/or contains errors in execution. Some work is used to find ratios, intercept, or equation or partial answers to portions of the task are evident.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them, and (2) Reason abstractly and quantitatively, since students need to abstract information from the graph, create a mathematical representation of the problem numerically or graphically, and consider whether the relationship is proportional. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem with tables, ratios and/or equations. Evidence of the Mathematical Practice, (6) Attend to precision, can include use of rise/run to extend the line and proper use of ratios. Evidence of the Mathematical Practice, (7) Look for and make use of structure, can include use of intentional techniques (rise/run) to extend the line and/or recognition that the equation of a proportional relationship is linear, with intercept equal to 0.

The reasoning used to solve the problem may include:

- The fact that the graph is a line automatically indicates proportionality.
- Some attempt to use slope is made, but fails to clearly explain how the slope can help determine proportionality.
- Some attempt to find the equation of the line is made, but not used to justify or refute proportionality in any way.



(2, 70)  
(3, 90)  
(4, 110)  
(5, 130)

This graph represents a proportional relationship because the x value increased by 1, while the y value increased by 20.

2. Kanye West expects to sell 350,000 albums in one week.
- How many albums will he have to sell every day in order to meet that expectation?
  - Kanye West has a personal goal of selling 5 million albums. If he continues to sell albums at the same rate, how long will it take him to achieve that goal? Explain your reasoning in words.
  - The equation  $y = 40,000x$ , where  $x$  is the number of days and  $y$  is the number of albums sold, describes the number of albums another singer expects to sell. Does this singer expect to sell more or fewer albums than West? Use mathematical reasoning to justify your response.

NYC Grade 8 Assessment 1  
Kanye West's Albums Task  
Benchmark Papers

3 Points

The response accomplishes the prompted purpose and effectively communicates the student's mathematical understanding. The student's strategy and execution meet the content (including concepts, technique, representations, and connections), thinking processes, and qualitative demands of the task. Minor omissions may exist, but do not detract from the correctness of the response.

Either verbally or symbolically, the strategy used to solve the problem is stated, as is the work used to find ratios, unit rates, and partial answers to problems. Minor arithmetic errors may be present, but no errors of reasoning appear. Complete explanations are stated based on work shown.

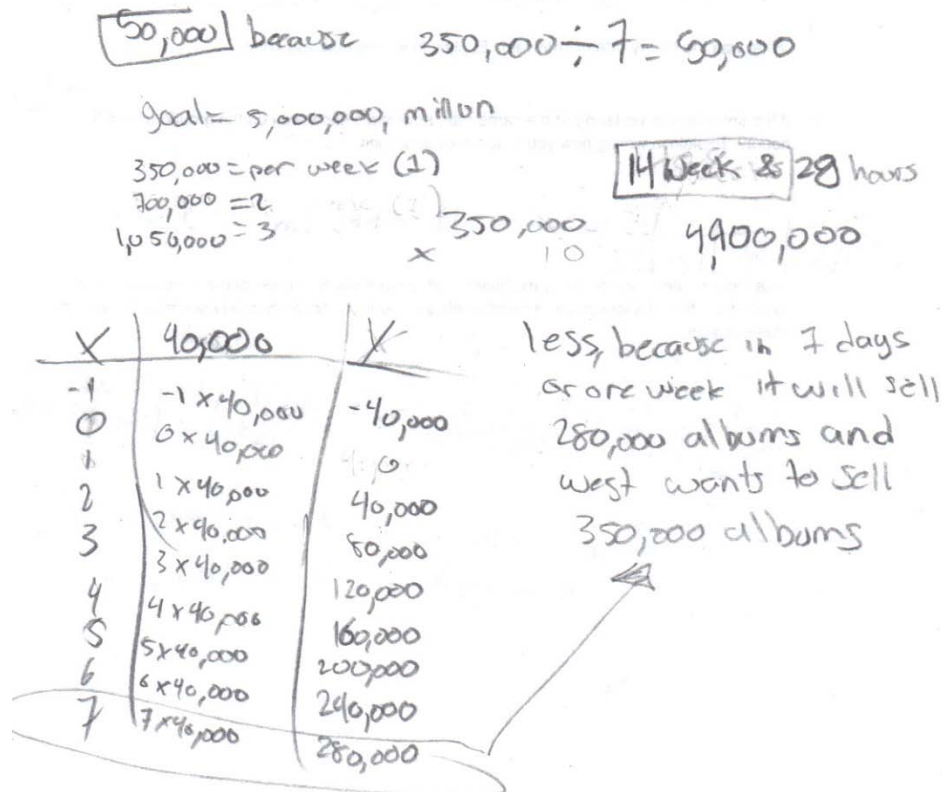
Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them and (2) Reason abstractly and quantitatively, since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with the ratio that names the probability). Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem as a ratio, decimal or percent in each portion of the task. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of ratio notation and proper labeling of quantities.

The reasoning used to justify part b may include:

- Scaling up from 50,000 to 5 million, after forming the ratio of 5 million : 50,000 or by using a table; possibly dividing the 50,000 albums per day unit rate into 5 million,
- Forming and solving the proportion  $50000/1 = 5,000,000/x$ , or  $350,000/7 = 5,000,000/x$  where  $x$  = number of days, possibly by scaling up or solving in the traditional manner.

The reasoning used to justify part c may include:

- Evaluating the second singer's number of albums sold per day as 40,000 and comparing that to Kanye West's 50,000.
- Evaluating the second singer's number of albums sold in a given number of days and correctly comparing that to the number sold by Kanye West in the same number of days.
- Finding the equation where  $y$  is the number of albums sold in  $x$  days for Kanye West ( $y = 50,000x$ ) and correctly comparing by using the slopes of the equations as unit rates.



NYC Grade 8 Assessment 1  
Kanye West's Albums Task  
Benchmark Papers

2 Points

The response demonstrates adequate evidence of the learning and strategic tools necessary to complete the prompted purpose. It may contain overlooked issues, misleading assumptions, and/or errors in execution. Evidence in the response demonstrates that the student can revise the work to accomplish the task with the help of written feedback or dialogue.

Either verbally or symbolically, the strategy used to solve the problem is stated, as is the work used to find ratios, unit rates, or partial answers to problems. Partial explanations are stated, based on work shown.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them, and (2) Reason abstractly and quantitatively, since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with the ratio that names the probability. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem as a ratio, decimal or percent in each portion of the task. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of ratio notation and proper labeling of quantities.

The reasoning used to justify part b may include:

- Attempting to scale up from 50,000 to 5 million, possibly by using a table, but failing to reach or stop at 5 million. Scaling up from 350,000 to 5 million, but failing to recognize that the result must be multiplied by the 7 days in the week.
- Forming and attempting to solve the proportion  $50000/1 = 5,000,000/x$ , or  $350,000/7 = 5,000,000/x$  where  $x$  = number of days, but using inappropriate processes to solve the proportion OR forming and correctly solving a proportion with an incorrect unit rate from part a.

The reasoning used to justify part c may include:

- Incorrectly evaluating the second singer's number of albums sold per day, then correctly comparing that number to Kanye West's 50,000.
- Evaluating the second singer's number of albums sold in a given number of days and incorrectly comparing that to the number sold by Kanye West in the same number of days.

$$\begin{array}{r} \times 50 \\ 7 \overline{) 350} \\ \underline{- 350} \\ 000 \end{array}$$

$$\begin{array}{r} 50000 \\ \times 7 \\ \hline 350000 \end{array}$$

He will have to sell 50,000 albums every day.

West will take 1,000 days to sell 50,000,000 albums. I divided 50,000 by 5 million because 50,000 is the amount of albums he sells everyday.

$$1 = 40,000x$$

$$1 = 40,000 \cdot 1,000$$

$$1 = 40,000,000$$

This artist will make less than West.

NYC Grade 8 Assessment 1  
Kanye West's Albums Task  
Benchmark Papers

1 Point

The response demonstrates some evidence of mathematical knowledge that is appropriate to the intent of the prompted purpose. An effort was made to accomplish the task, but with little success. Evidence in the response demonstrates that, with instruction, the student can revise the work to accomplish the task.

Some evidence of reasoning is demonstrated either verbally or symbolically, but may be based on misleading assumptions, and/or contain errors in execution. Some work is used to find ratios, or unit rates; or partial answers to portions of the task are evident. Explanations are incorrect, incomplete or not based on work shown.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them and (2) Reason abstractly and quantitatively, since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with the ratio that names the probability. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem as a ratio, decimal, or percent in each portion of the task. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of ratio notation and proper labeling of quantities.

The reasoning used to solve the parts of the problem may include:

- Some attempt to scale.
- Some attempt to form a proportion.
- Failure to attempt at least two parts of the problem or failure to attempt some kind of explanation in parts b and c.

$$\begin{array}{r} 50000 \\ 7 \overline{) 350000} \end{array} \quad \boxed{50,000}$$

it would take him 14.5 weeks  
to reach 5 million albums sold

$$\begin{aligned} h & y = 40,000x \\ x & = \text{days} \\ y & = \text{albums sold} \end{aligned}$$

3. Marvin likes to run from his home to the recording studio. He uses his iPod to track the time and distance he travels during his run. The table below shows the data he recorded during yesterday's run.

Time (Minutes)	Distance (KM)
5	0.833
10	1.660
15	2.545
20	3.332
25	4.003
30	5.012
35	5.831

- a) Write an algebraic equation to model the data Marvin collected. Explain, in words, the reasoning you used to choose your equation.
- b) Does the data represent a proportional relationship? Explain your reasoning in words.
- c) If Marvin continues running at the pace indicated in your equation, how long will it take him to reach the recording studio, which is 12 km from his home? Use mathematical reasoning to justify your response.

NYC Grade 8 Assessment 2  
Marvin's Run  
Rubrics and Benchmark Papers

3 Points

The response accomplishes the prompted purpose and effectively communicates the student's mathematical understanding. The student's strategy and execution meet the content (including concepts, technique, representations, and connections), thinking processes, and qualitative demands of the task. Minor omissions may exist, but do not detract from the correctness of the response.

Either verbally or symbolically, the strategy used to solve each part of the problem is stated, as is the work used to find ratios, slope, equation, and partial answers to problems. Minor arithmetic errors may be present, but no errors of reasoning appear. While responses to the prompt, "Explain your reasoning in words", must include words and sentences containing a line of reasoning appropriate to the problem, responses to the prompt, "Use mathematical reasoning to justify your response", must include appropriate mathematical symbolism and/or words.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them, and (2) Reason abstractly and quantitatively, since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with slope and intercept. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by the linear equation shown in part a. Evidence of the Mathematical Practice, (5) Use appropriate tools strategically, may be demonstrated by use of the graphing calculator to "see" the data graphically, draw a sketch of the graph, and use it to solve part a. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of rate, proper symbolism, and proper labeling of quantities and graphs. Evidence of the Mathematical Practice, (7) Look for and make use of structure, may be demonstrated identifying the relationship as proportional because it follows the general  $d = rt$  pattern.

The response contains correct explanations for all parts, as required.

The reasoning used to solve part a may include:

- A graph of the data (or a sketch from data displayed on a graphing calculator); drawing a single line through the data on the graph that attempts to account for as much of the data as possible. Likely lines include: one such that some points appear above, some below and some directly on the line; OR one that passes through the left- and right-most points on the graph, etc.
- Using the line drawn through the points, forming similar triangles, drawing steps through points on the line, or correctly forming the ratio of the difference between two y-values on the line to the difference between two x-values on the line, etc., to find the slope of the line drawn.
- If their line passes through (0, 0), acknowledging (0, 0) as the initial value and choosing the 0 to be the y-intercept, OR using the y-value of the point where their line intercepts the y-axis as their y-intercept.
- Using the values determined above to form a linear equation of the form  $y = mx + b$  (or some equivalent model, e.g. in point-slope form).
- Possibly determining  $5.831/35 = 0.1666... \approx 0.17$  as the slope, since it represents the average speed, and choosing 0 as the y-intercept, especially if accompanied by an explanation that the relationship is a proportional one,  $d = rt$ , where  $d$  = distance in km,  $r$  = rate in km/min. and  $t$  = time in minutes.

The reasoning used to solve part b may include:

- Identifying the relationship as proportional because their choice of line appears to pass through (0, 0) indicating that the relationship is the proportional one,  $d = rt$ , where  $d$  = distance in km,  $r$  = rate in km/min. and  $t$  = time in minutes, or noting that each y-value is 0.17 times the x-value.
- Identifying the relationship as non-proportional because their choice of line does NOT pass through (0, 0).

The reasoning used to solve part c may include:

- Solving their equation for  $y = 12$ , e.g.,  $0.17x = 12$ .
- Extending the table or graph to 60 minutes, using the slope chosen for part a.

NYC Grade 8 Assessment 2  
Marvin's Run  
Rubrics and Benchmark Papers

2 Points

The response demonstrates adequate evidence of the learning and strategic tools necessary to complete the prompted purpose. It may contain overlooked issues, misleading assumptions, and/or errors in execution. Evidence in the response demonstrates that the student can revise the work to accomplish the task with the help of written feedback or dialogue.

Either verbally or symbolically, the strategy used to solve the problem is stated, as is the work used to find slope, intercept, equation, and prediction in part c. Reasoning may contain incomplete, ambiguous, or misrepresentative ideas. Partial explanations are stated, based on work shown.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them, and (2) Reason abstractly and quantitatively, since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with slope and intercept. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by the linear equation shown in part a. Evidence of the Mathematical Practice, (5) Use appropriate tools strategically, may be demonstrated by use of the graphing calculator to "see" the data graphically, draw a sketch of the graph, and use it to solve part a. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of rate, proper symbolism, and proper labeling of quantities and graphs. Evidence of the Mathematical Practice, (7) Look for and make use of structure, may be demonstrated identifying the relationship as proportional because it follows the general  $d = rt$  pattern.

The response contains correct explanations for most parts, as required, OR correct calculations to all parts with few or no explanations.

The reasoning used to solve part a of the problem may include:

- a. Drawing a single line through the data on the graph that attempts to account for as much of the data as possible, determining the slope, but failing to choose appropriate scale on the x-axis, possibly using {1, 2, 3, 4, 5, etc.} instead of {5, 10, 15, 20, 25, etc.}

OR

- b. Drawing an arbitrary single line through the data; then, using the line drawn through the points, forming similar triangles, drawing steps through points on the line, or correctly forming the ratio of the difference between two y-values on the line to the difference between two x-values on the line, etc., to find the slope of the line drawn.

OR

- c. Determining the slope logically, but choosing the y-intercept arbitrarily.

OR

- d. Possibly determining  $5.831/7 \approx 0.83$  as the slope, accounting for 7 intervals, but not the 35 minutes, and choosing 0 as the y-intercept, especially if accompanied by an explanation that the relationship is a proportional one,  $d = rt$ , where  $d$  = distance in km,  $r$  = rate in km/min. and  $t$  = time in minutes.

The reasoning used to solve part b may include:

- a. Graphing a line passing through (0, 0) and identifying the relationship as proportional but providing no rationale as to why.
- b. Graphing a line NOT passing through (0, 0) and identifying the relationship as non-proportional but providing no rationale as to why.

The reasoning used to solve part c may include failing to acknowledge the problem's prompt, "...at pace indicated in your equation," and:

- a. Attempting to solve their equation for  $y = 12$ , e.g.,  $0.17x = 12$ , but not being able to handle the decimal coefficient properly.
- b. Unsuccessfully attempting to extend the table or graph to 60 minutes, using the slope chosen for part a.
- c. Doubling, e.g., the 30-minute value, 5.012 to 10.024, since the relationship is proportional (or some variation of this method).

NYC Grade 8 Assessment 2  
Marvin's Run  
Rubrics and Benchmark Papers

- a) Write an algebraic equation to model the data Marvin collected. Explain, in words, the reasoning you used to choose your equation.

$$0.1666x = y$$

- b) Does the data represent a proportional relationship? Explain your reasoning in words.

You represent a proportional relationship  
because you multiply 0.1666 to the  
minutes

- c) If Marvin continues running at the pace indicated in your equation, how long will it take him to reach the recording studio, which is 12 km from his home? Use mathematical reasoning to justify your response.

72 minutes because if you divide  
12 by 0.1666 which gives 72

NYC Grade 8 Assessment 2  
Marvin's Run  
Rubrics and Benchmark Papers

1 Point

The response demonstrates some evidence of mathematical knowledge that is appropriate to the intent of the prompted purpose. An effort was made to accomplish the task, but with little success. Evidence in the response demonstrates that, with instruction, the student can revise the work to accomplish the task.

Some evidence of reasoning is demonstrated either verbally or symbolically, is often based on misleading assumptions, and/or contains errors in execution. Some work is used to find ratios and probability or partial answers to portions of the task are evident. Explanations are incorrect, incomplete, or not based on work shown. Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them and (2) Reason abstractly and quantitatively, since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with slope and intercept. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by the linear equation shown in part a. Evidence of the Mathematical Practice, (5) Use appropriate tools strategically, may be demonstrated by use of the graphing calculator to "see" the data graphically, draw a sketch of the graph, and use it to solve part a. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of rate, proper symbolism, and proper labeling of quantities and graphs. Evidence of the Mathematical Practice, (7) Look for and make use of structure, may be demonstrated identifying the relationship as proportional because it follows the general  $d = rt$  pattern.

Responding correctly to only one part of the problem; for example, two responses are similar to those described below.

The reasoning used to solve the part a of the problem may include doing two or more of the following:

- a. Drawing an arbitrary single line through the data, then, using the line drawn through the points, forming similar triangles, drawing steps through points on the line, or correctly forming the ratio of the difference between two y-values on the line to the difference between two x-values on the line, etc., to find the slope of the line drawn.
- b. Choosing the y-intercept to be 0, regardless of whether or not their line passes through that point;
- c. Using the values determined above to form a linear equation of the form  $y = mx + b$  (or some equivalent model, e.g. in point-slope form), but switching the m and b.

The reasoning used to solve part b of the problem may include:

- a. Graphing a line passing through (0, 0) and identifying the relationship as non-proportional but providing no rationale as to why.
- b. Graphing a line NOT passing through (0, 0) and identifying the relationship as proportional but providing no rationale as to why.

The reasoning used to solve part c of the problem may include:

- a. Solving their equation for  $x = 12$ , instead of  $y = 12$ .
- b. Extending the table or graph to 60 minutes by continually adding 0.819 (the difference between the last two y-values) or some other difference between y-values.

NYC Grade 8 Assessment 2  
Marvin's Run  
Rubrics and Benchmark Papers

- a) Write an algebraic equation to model the data Marvin collected. Explain, in words, the reasoning you used to choose your equation.

$$0.833 \div 5 \approx 0.1666$$

I chose my equation so I can figure out what's = to 1 min

- b) Does the data represent a proportional relationship? Explain your reasoning in words.

yes because the numbers increases

- c) If Marvin continues running at the pace indicated in your equation, how long will it take him to reach the recording studio, which is 12 km from his home? Use mathematical reasoning to justify your response.

$$1.9992 \quad 0.1666 \times 12 = 1.9992$$

The time it would take would be 13 min

4. Jumel and Ashley have two of the most popular phones on the market, a Droid and an iPhone. The cost of both monthly cell phone plans are described below.
- Jumel's plan:  $c = 60 + 0.05t$ , where  $c$  stands for the monthly cost in dollars, and  $t$  stands for the number of texts sent each month.
  - Ashley's plan: \$.35 per text, in addition to a monthly fee of \$45.
- a. Whose plan, Jumel's or Ashley's, costs less if each of them sends 30 texts in a month? Use mathematical reasoning to justify your answer.
  - b. How much will Ashley's plan cost for the same number of texts as when Jumel's plan costs \$75.00?
  - c. Is there a number of texts for which both plans cost the same amount? Use mathematical reasoning to justify your answer.

NYC Grade 8 Assessment 1  
Droid versus iPhone Task  
Benchmark Papers

3 Points

The response accomplishes the prompted purpose and effectively communicates the student's mathematical understanding. The student's strategy and execution meet the content (including concepts, technique, representations, and connections), thinking processes, and qualitative demands of the task. Minor omissions may exist, but do not detract from the correctness of the response.

Either verbally or symbolically, the strategy used to solve each part of the problem is stated, as is the work used to evaluate costs, solve equations, graph, and find partial answers to problems. Minor arithmetic errors may be present, but no errors of reasoning appear.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them, and (2) Reason abstractly and quantitatively, since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with the tables, graphs or equations. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem as tables, graphs or equations. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of ratio notation and proper labeling of quantities, variables and graphs. Evidence of the Mathematical Practice, (7) Look for and make use of structure, may be demonstrated by student recognition that each part of the task can be answered by a single representation.

The reasoning used to solve the parts of the problem may include:

- Evaluating expressions to answer part a.
- Expressing Ashley's plan in equation form, and using equations to answer some or all of the three parts.
- Setting up tables and expanding to answer some or all of the three parts.
- Graphing both plans on the same set of axes to answer some or all of the three parts.
- Using slope and intercept explanations, possibly with a sketch, to answer part c.

a.  $C = 60 + 30 \times 0.05$  |  $C = 45 + 35 \times 30$   
 $C = 61.5$  |  $C = 55.5$   
 June | Ashley

b.  $45 + 300 \times .35$  June = 6 dollars more  
 Her plan would cost \$150.  $\frac{6}{.05} = 120$  more  
 $69.00 = \text{Ashley}$   $0.05 \times 300 = 15$

c. If they both sent 50 text messages then their plans would cost the same.

Ashley	June
$C = 45 + .35t$	$C = 60 + .05t$
$45 + .35t = 60 + .05t$	
$-.05t$	$-.05t$
$45 + .30t = 60$	
$-45$	$-45$
$.30t = 15$	
$\div .30$	$\div .30$
$t = 50$	

NYC Grade 8 Assessment 1  
Droid versus iPhone Task  
Benchmark Papers

2 Points

The response demonstrates adequate evidence of the learning and strategic tools necessary to complete the prompted purpose. It may contain overlooked issues, misleading assumptions, and/or errors in execution. Evidence in the response demonstrates that the student can revise the work to accomplish the task with the help of written feedback or dialogue.

Either verbally or symbolically, the strategy used to solve each part of the problem is stated, as is the work used to evaluate costs, solve equations, graph, and find partial answers to problems. Minor arithmetic errors may be present. Reasoning may contain incomplete, ambiguous, or misrepresentative ideas.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them and (2) Reason abstractly and quantitatively, since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with the tables, graphs or equations. Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem as tables, graphs, or equations. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of ratio notation and proper labeling of quantities, variables, and graphs. Evidence of the Mathematical Practice, (7) Look for and make use of structure, may be demonstrated by student recognition that each part of the task can be answered by a single representation.

At least two of the three parts is correctly completed. Justification may include reasoning as follows:

- Incorrectly evaluating expressions to answer part a, but correctly indicating the less expensive plan, based on the calculations.
- Incorrectly expressing Ashley's plan in equation form, but correctly using the resulting equations to answer some or all of the three parts.
- Incorrectly setting up tables and expanding them, but correctly answering some or all of the three parts based on the calculations.
- Incorrectly graphing both plans on the same set of axes, but correctly answering some or all of the three parts based on the graphs.
- Using slope and intercept explanations with errors, possibly with a sketch, but correctly answering part c based on the calculations.

a. *Timel*  $\rightarrow$  
$$\begin{array}{r} 0.05 \\ \times 30 \\ \hline 1.50 \\ + 60 \\ \hline 61.50 \end{array}$$
 *Ashley*  $\rightarrow$  
$$\begin{array}{r} 0.35 \\ \times 30 \\ \hline 10.50 \\ + 45 \\ \hline 55.50 \end{array}$$
 Ashley's plan cost less

b. 
$$\begin{array}{r} 75 \\ - 61.50 \\ \hline 13.5 \\ + 0.05 \\ \hline 13.55 \\ + 30 \\ \hline 43.55 \end{array}$$
 
$$\begin{array}{r} 0.35 \\ \times 30 \\ \hline 10.5 \\ + 45 \\ \hline 55.5 \\ + 150 \\ \hline 205.5 \end{array}$$
 \$150

c. 
$$\begin{array}{r} 0.35 \\ \times 50 \\ \hline 17.50 \\ + 45 \\ \hline 62.50 \end{array}$$
 
$$\begin{array}{r} 0.05 \\ \times 50 \\ \hline 2.50 \\ + 60 \\ \hline 62.50 \end{array}$$

NYC Grade 8 Assessment 1  
Droid versus iPhone Task  
Benchmark Papers

1 Point

The response demonstrates some evidence of mathematical knowledge that is appropriate to the intent of the prompted purpose. An effort was made to accomplish the task, but with little success. Evidence in the response demonstrates that, with instruction, the student can revise the work to accomplish the task.

Some evidence of reasoning is demonstrated either verbally or symbolically, but may be based on misleading assumptions, and/or contain errors in execution. Some work is used to evaluate costs, solve equations, or graph.

Accurate reasoning processes demonstrate the Mathematical Practices, (1) Make sense of problems and persevere in solving them, and (2) Reason abstractly and quantitatively (since students need to abstract information from the problem, create a mathematical representation of the problem, and correctly work with the tables, graphs or equations). Evidence of the Mathematical Practice, (3) Construct viable arguments and critique the reasoning of others, is demonstrated by complete and accurate explanations. Evidence of the Mathematical Practice, (4) Model with mathematics, is demonstrated by representing the problem as tables, graphs or equations. Evidence of the Mathematical Practice, (6) Attend to precision, can include proper use of ratio notation and proper labeling of quantities, variables and graphs. Evidence of the Mathematical Practice, (7) Look for and make use of structure, may be demonstrated by student recognition that each part of the task can be answered by a single representation.

At least one of the three parts is correctly completed. Justification may include reasoning as follows:

- Incorrectly evaluating expressions to answer part a, but correctly indicating the less expensive plan, based on the calculations.
- Incorrectly expressing Ashley's plan in equation form, but correctly using the resulting equations to answer some or all of the three parts.
- Incorrectly setting up tables and expanding them, but correctly answering some or all of the three parts based on the calculations.
- Incorrectly graphing both plans on the same set of axes, but correctly answering some or all of the three parts based on the graphs.
- Using slope and intercept explanations with errors, possibly with a sketch, but correctly answering part c based on the calculations.

a. J;  $C = 60 + 0.05 \cdot 30$   
 $C = 60 + 1.5$   
 $C = 61.50$

A;  $C = 45 + .35 \cdot 30$   
 $C = 45 + 10.5$   
 $C = 55.50$

b. 150 text.  
 $45 + .35 \cdot 150$   
 $45 + 52.5$   
 $97.50$

Ashley's Plan costs less r.  
 Ashley's Plan will cost 97.50 dollars.

c. I know they cost the same amount because they both sent text until they reached \$75.00 for that month.



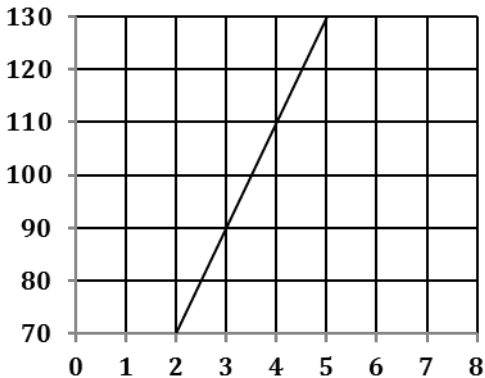
## GRADE 8 MATH: EXPRESSIONS & EQUATIONS

### ANNOTATED STUDENT WORK

This section contains annotated student work at a range of score points for questions 1,2 and 5. For student work samples for questions 3 and 4, please refer to the benchmark papers in the rubric section.

## Assessment 1: Question 1

1. Does the graph below represent a proportional relationship? Use mathematical reasoning to justify your response.



CCSS (Content) Addressed by this Task:

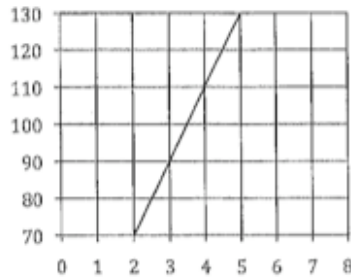
- 7.RP.2 Recognize and represent proportional relationships between quantities.
- 7.RP.2a Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
- 8.EE.6 (potential) Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for a line intercepting the vertical axis at  $b$ .

CCSS for Mathematical Practice Addressed by the Task:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
6. Attend to precision
7. Look for and make use of structure

NYC Grade 8 Assessment 1  
Determining Proportionality Task  
Annotated Student Work

Annotation of Student Work With a Score of 3



No, because they are not in proportion  
Cross products  $\neq$

$$\frac{3}{90} \neq \frac{4}{110}$$

$$\frac{4}{110} \neq \frac{5}{130}$$

$$\frac{3}{90} \neq \frac{5}{130}$$

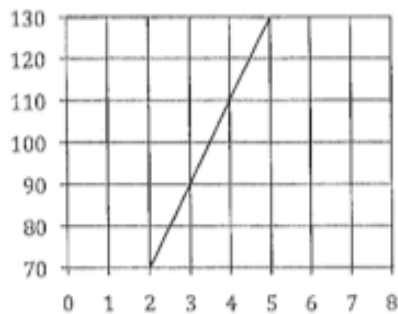
**Content Standards:** The student receives a score of 3 because the work shows ratios that are formed using coordinates that are easily read from the graph (7.RP.2a) The student indicates that the ratios are not equivalent and uses this fact to conclude that the graph does not represent a proportional relationship since “they are not in proportion” (7.RP.2a)

**Mathematical Practices:** The student shows the ability to model the situation mathematically with ratio comparisons (Practice 4), while in the process of developing a solution strategy and carrying it out successfully (Practice 1). In particular, the student shows the ability to analyze a relationship given in a graphical representation and translate this to a numerical representation in order to make a decision about the nature of the relationship (Practices 2 & 3). The work is made clear by the use of ratios that have been accurately determined from the graph, and proper mathematical notation has been used to indicate that the ratios are not equivalent (Practice 6). The student’s work indicates an understanding of the multiplicative structure of a proportion since s/he recognizes when this structure is missing (Practice 7).

**Next Instructional Steps:** The student argues, “No, because they are not in proportion.” Cross products  $\neq$ , leaving his/her reader to analyze the inequalities s/he presents to see what is meant. Give this student a strong “argument” response from another student to analyze. Ask the student to compare and contrast his/her own response with the other student’s response. Challenge the student to revise his/her response to add clarity and depth, and then to write a learning reflection on the process. Also ask the student, “How many pairs of ratios do you need to test for equivalence before you can conclude that a linear graph does or does not represent a proportional relationship? How else can you decide if a relationship is proportional if you only have a graph of the relationship?”

NYC Grade 8 Assessment 1  
Determining Proportionality Task  
Annotated Student Work

**Annotation of Student Work With a Score of 2**



$$\frac{5}{130} \quad \frac{4}{110} \quad \frac{3}{90} \quad \frac{2}{80}$$

No because if you take any two coordinates they are not equal

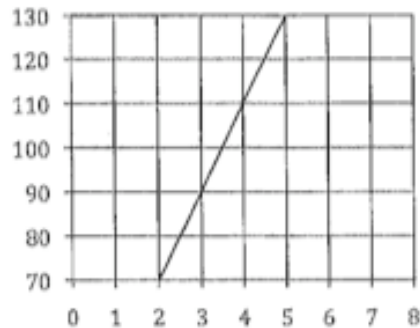
**Content Standards:** The student receives a score of 2 because the work shows ratios that are formed using coordinates that are easily read from the graph (7.RP.2a). The student's written explanation indicates that s/he recognizes the relationship is not proportional (7.RP.2).

**Mathematical Practices:** Forming ratios from the coordinates demonstrates the student is making sense of the problem (Practice 1) and persevering to answer the question. Forming ratios from the coordinates is also an indication that the student can reason abstractly and quantitatively (Practice 2) since s/he was able to analyze the graph and translate this to a numerical representation. Although the student's written explanation indicates that s/he recognizes the relationship is not proportional, the explanation is not mathematically accurate and no justification is given for the claim that "any two coordinates are not equal." (Weak response on Practice 3.) The lack of clarity in the explanation is partially due to weakness on Practice 6, since the student refers to "coordinates" rather than ratios of coordinates. Weakness on MP 7 is indicated since the student fails to describe what structure is being observed when s/he claims, "any two coordinates are not equal."

**Next Instructional Steps:** Since the student is weak on accurate mathematical descriptions, model precise mathematical language by asking the student to demonstrate how s/he knows the ratios are not equivalent, and why this means the relationship is not proportional. Ask the student to rephrase and refine his/her answer to make it more accurate.

NYC Grade 8 Assessment 1  
Determining Proportionality Task  
Annotated Student Work

**Annotation of Student Work With a Score of 1**



No this graph is not Proportional  
because the line does not Pass ~~thru~~  
through the origin

**Content Standards:** This student receives a score of 1 because s/he correctly claims that the “graph is not proportional” and the reason stated is accurate (7.RP.2). What is lacking in the student’s response is any indication as to whether not s/he has verified or how s/he has determined that the line “does not pass through the origin.”

**Mathematical Practices:** The student’s response demonstrates s/he is making sense of the problem (Practice 1). The lack of justification for the claim that the line “does not pass through the origin” indicates a weakness on Practices 2 and 3. There is little evidence of Practice 6 or 7 since the student has not extended the line or acknowledged what structures are being used to support his/her claim. In addition the student’s statement that the “graph is not proportional” is not mathematically precise (Practice 7) in that it is a “relationship between quantities” that may be described as proportional. The graph is a representation of a relationship and not the relationship itself, so a more appropriate statement would be, “the quantities represented in the graph are in a proportional relationship to each other.”

**Next Instructional Steps:** Ask the student how s/he knows the line does not pass through the origin and why this is a test for proportionality.

## Assessment 1: Question 2

2. Kanye West expects to sell 350,000 albums in one week.
- How many albums will he have to sell every day in order to meet that expectation?
  - Kanye West has a personal goal of selling 5 million albums. If he continues to sell albums at the same rate, how long will it take him to achieve that goal? Explain your reasoning in words.
  - The equation  $y = 40,000x$ , where  $x$  is the number of days and  $y$  is the number of albums sold, describes the number of albums another singer expects to sell. Does this singer expect to sell more or fewer albums than West? Use mathematical reasoning to justify your response.

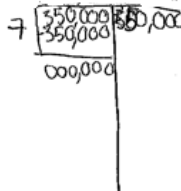
### CCLS (Content) Addressed by this Task:

- |                     |   |
|---------------------|---|
| 8.EE.5<br>(partial) | Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. |
| 7.RP.2              | Recognize and represent proportional relationships between quantities.  |
| 7.RP.3              | Use proportional relationships to solve multi-step ratio and percent problems.  |
| 8.F.2               | Compare properties of two functions each represented in a different way.  |

### CCLS for Mathematical Practice Addressed by the Task:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
6. Attend to precision

**Annotation of Student Work With a Score of 3**

- a.  He would have to sell 50,000 albums a day
- b. It would take him 100 days because if he sells 50,000 albums a day and 5 million divided by 50,000 is 100 then it would take 100 days.
- c.  $4,000,000 = 40,000 \times 100$   
The singer expects to sell fewer albums than West because if in 100 days West sells 5,000,000 albums and in 100 days the artist sells 4,000,000 albums, then they would have expected to sell fewer albums than West.

**Content Standards:** The student receives a score of 3 because s/he divides 350,000 albums sold in one week by 7 to determine the number of albums sold in 1 day, thus finding the unit rate in number of albums per day (7.RP.2). In part b) the student uses the unit rate of "50,000 albums a day", correctly divides 5,000,000 by 50,000 and goes on to state that it will "take 100 days" (7.RP.3). In part c), the student evaluates the number of albums sold by the second singer in 100 days and correctly compares that quantity to the number of albums sold by Kanye West in the same number of days. The student concludes that the second artist will "sell fewer albums than West." Since West's sales rate is given verbally and the second singer's is given as an equation, the student demonstrates evidence of understanding 8.EE.5 and 8.F.2.

**Mathematical Practices:** The student shows the ability to model the situation mathematically (Practice 4) while in the process of developing a solution strategy and carrying it out successfully (Practice 1). The student demonstrates the ability to abstract the given situation and represent it numerically; then, the student manipulates the numbers and translates back again to the context (Practice 2). The work is made clear by the use of labels for the quantities (Practice 6). Finally, the student gives complete explanations of his/her reasoning and explains the meaning of his/her results (Practice 3).

**Next Instructional Steps:** Since the student uses both unit rates in his/her solution but does not identify them as such, ask the student if s/he can identify one or more unit rates in this problem situation and what role the rate/s play in the context.

**Annotation of Student Work With a Score of 2**

a 7 days

$$\begin{array}{r} 50000 \cdot \\ 350000 \overline{) 7} \\ 00 \\ 00 \\ 00 \\ 00 \end{array}$$
 he will have  
to see 50,000 albums every day

b

$$\begin{array}{r} 50000 \\ 100 \overline{) 5000000} \\ 500000 \\ \hline 5000000 \end{array}$$
 he will take 100 day to complete his goal.

because he sell 50,000 a day and he will sell 5 million in 100 days.

c They would sell less because 50,000 is more then 40,000.

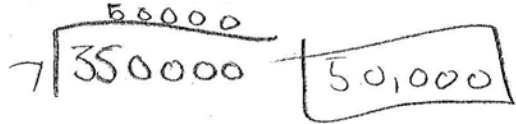
**Content Standards:** The student receives a score of 2 because the response demonstrates adequate evidence of the learning and strategic tools necessary to answer the questions in parts a) and b). A partial explanation is given in response to part c). In part a), the student divides 350,000 by 7 to determine the number of albums sold in 1 day, thus finding the unit rate in number of albums per day (7.RP.2). In part b) the student scales the 50,000 to 5,000,000 using a scale factor of 100 (7.RP.3), although it is not clear how the student determines that 100 is the scale factor needed in this situation. The student correctly states, "he will sell 5 million in 100 days." In part c), the student's response is incomplete since s/he appears to have lost track of one of the quantities needed for the comparison, i.e., albums: days.

**Mathematical Practices:** The student shows the ability to partially model the situation mathematically (Practice 4) in the process of developing a solution strategy and carrying it through, although not completely successfully (Practice 1). The lack of clarity and a lack sound justification (Practice 3) in part c) is partially due to weakness on Practice 6, since the student writes only about the number of albums without reference to the number of days. In parts a) and b), the student demonstrates the ability to abstract the given situation and represent it numerically; then, the student manipulates the numbers and translates back again to the context (Practice 2).

**Next Instructional Steps:** Ask the student to read back his/her response to part c) and ask what these numbers mean in the context of the problem and why this matters.

NYC Grade 8 Assessment 1  
Kanye West's Albums Task  
Annotated Student Work

**Annotation of Student Work With a Score of 1**

- a. 
- b. it would take him 14.5 weeks  
to reach 5 million albums sold
- c.  $h = y = 40,000x$   
 $x = \text{days}$   
 $y = \text{albums sold}$

**Content Standards:** The student receives a score of 1 because s/he demonstrates some evidence of mathematical knowledge that is appropriate for some parts of the question. In part a), the student correctly divides to determine the number of albums sold in 1 day (7.RP.2), although none of the quantities are labeled with the appropriate units. In part b), the student's answer is reasonable for the amount of time it will take for West to sell 5 million albums (7.RP.3); however, the student gives no indication as to how 14.5 weeks is determined. The student makes no attempt to solve part c), but merely summarizes what is given.

**Mathematical Practices:** The student's work indicates that s/he is attempting to make sense of the problem and persevere since something is written for all 3 parts of the problem with partial success in answering the questions (Practice 1). The student shows partial ability with Practice 2 in part a), since s/he is able to abstract the given situation, represent it numerically, and manipulate the numbers. The student, however, fails to translate the numbers back into the context in part a). The work is weak on Practice 3 since, in part b), no indication is given as to how 14.5 weeks is determined. The student demonstrates limited ability with Practice 4, since s/he chooses the appropriate operation in part a) and gives a reasonable contextual answer, albeit unsubstantiated, in part b). The student shows weakness on Practice 6 since the quantities in part a) are without labels.

**Next Instructional Steps:** Give this student a strong response from another student to analyze. Ask the student to compare and contrast his/her own response with the other student's response. Challenge the student to revise his/her response to add clarity and depth, and then to write a learning reflection on the process.

## Assessment 1: Question 5

5. Jumel and Ashley have two of the most popular phones on the market, a Droid and an iPhone. The cost of both monthly cell phone plans are described below.

▫ Jumel's plan:  $c = 60 + 0.05t$ , where  $c$  stands for the monthly cost in dollars, and  $t$  stands for the number of texts sent each month.

▫ Ashley's plan: \$.35 per text, in addition to a monthly fee of \$45.

- Whose plan, Jumel's or Ashley's, costs less if each of them sends 30 texts in a month? Use mathematical reasoning to justify your answer.
- How much will Ashley's plan cost for the same number of texts as when Jumel's plan costs \$75.00?
- Is there a number of texts for which both plans cost the same amount? Use mathematical reasoning to justify your answer.

CCLS (Content) Addressed by this Task:

- 8.EE.7 Solve linear equations in one variable.
- 8.EE.8 Analyze and solve pairs of simultaneous linear equations.
- 8.EE.8c Solve real-world and mathematical problems leading to two linear equations in two variables.
- 8.F.2 Compare properties of two functions each represented in a different way.  
Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of the relationship or from two  $(x, y)$  values. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
- 8.F.4

CCLS for Mathematical Practices Addressed by the Task:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
6. Attend to precision

NYC Grade 8 Assessment 1  
Droid versus iPhone Task  
Annotated Student Work

Annotation of Student Work With a Score of 3

- a.
- $$J = C = 60 + .05t$$
- $$A = C = 45 + .35t$$
- Figured if  $t = 30$   
then Ashley's is 35.50 less  
and Jumel's is 61.50
- b.
- $$\begin{array}{r} 45.00 \\ + 105.00 \\ \hline \$150.00 \end{array}$$
- $$\begin{array}{r} 75.00 \\ - 60.00 \\ \hline \$15.00 \\ \hline .05 \\ \hline 300 = t \end{array}$$
- c.
- if  $t = 20$  then  $J = \$61$   
 $A = \$87$
- $$\begin{array}{r} 40 \\ A = 45 + (.35 \cdot 40) = \$59 \\ J = 60 + (.05 \cdot 40) = \$62 \end{array}$$
- $$\begin{array}{r} 50 \\ 45 + (.35 \cdot 50) = \$62.50 \\ 60 + (.05 \cdot 50) = \$62.50 \end{array}$$

**Content Standards:** The student receives a score of 3 because the student responds with appropriate mathematical reasoning in all three parts of the problem. In part a), the student has constructed a linear function to model Ashley's plan (8.F.4). The student correctly determines the cost of 30 texts for both Jumel and Ashley and indicates Ashley's plan costs less (8.F.2). In part b), the student uses the equation for Jumel's plan to determine the number of texts Jumel can make for \$75 which s/he writes as  $300 = t$  (8.EE.7). The student's work in the left portion of his/her response in part b) seems to indicate that the student substitutes  $t = 300$  into the equation s/he wrote for Ashley's plan in part a), since  $.35(300) = 105$  and arrives at the correct answer of \$150 for Ashley's cost (8.EE.8c). In part c), the student evaluates the costs for both Ashley and Jumel for  $t = 20$  and  $t = 40$ , substituting  $t = 40$  into both equations and determines the cost is not equivalent. S/he then substitutes  $t = 50$  and determines that the cost for both plans is the same (8.EE.8 & 8.EE.8c).

**Mathematical Practices:** The student makes sense of the problem (Practice 1) and is able to model the situation abstractly and quantitatively and relate the answers back to the context (Practices 2 and 4). Justification for his/her conclusions can strengthen the student's work (Practice 3), especially in part c). Although the student's calculations are accurate and s/he has labeled many of the quantities appropriately with dollar signs, the student's failure to verbally communicate his/her reasoning indicates weakness on Practice 6.

**Next Instructional Steps:** Group the student with other students who took a different approach to the problem, especially in part c). Ask the students to discuss and compare the different strategies. Afterwards, challenge the students to explain, in writing why their method works and how it is the same and different from at least one other strategy.

NYC Grade 8 Assessment 1  
Droid versus iPhone Task  
Annotated Student Work

Annotation of Student Work With a Score of 2

- a.  $\$55.50$  because  $30 \times .35 = 10.50$   $\$61.50$
- b.  $\$75.00$   
 $-\$60.00$   
 $\$15.00$   
 $\downarrow$   
she make 300 texts  
 $15.00$   
 $9.00$   
 $105.00 \div 45 = \$150$
- c. amount:  $66.00$   $60t$   $66.00$   $67.50$   
 $60$   $10.50$   $Ashley's$   $150t$   $Jumel's$   
 $10.50$   $plan$   $plan$   
 $H = 45 + (35.45) = 60.45$   
 $J = 60 + (45.1)$

**Content Standards:** The student receives a score of 2 because the student responds with appropriate mathematical reasoning in two of the three parts of the problem. In part a), the student uses multiplication with the number of texts, 30, and the rate per text for both plans and correctly evaluates the cost on both plans for 30 texts (8.F.2). In part b), the student uses the equation for Jumel's plan to determine the number of texts Jumel can make for \$75 (8.EE.7) and s/he correctly states, "she makes 300 texts." The student uses this result and multiplies it by Ashley's rate per text of \$.35, then adds the monthly fee to arrive at \$150 (8.EE.8c). In part c), it appears that the student erroneously decides to evaluate the cost of Ashley's plan for 60 texts, "60t", to arrive at "66.00." The student evaluates the cost of Jumel's plan for 150 texts, "150t", to arrive at "67.50." The student fails to state a conclusion in part c).

**Mathematical Practices:** The student makes sense of the problem (Practice 1) and is able to model the situation abstractly and quantitatively and relate the answers back to the context (Practices 2 and 4). Justification for his/her conclusions can strengthen the student's work (Practice 3) in all parts of the problem. Although the student's calculations are accurate in parts a) and b) and s/he has labeled many of the quantities appropriately with dollar signs, the student's failure to verbally communicate his/her reasoning indicates weakness on Practice 6.

**Next Instructional Steps:** Ask the student what his/her strategy is in part c) to find if there is some number of texts for which both plans are the same amount. Ask the student how 60t for Ashley's plan and 150t for Jumel's plan is helping him/her decide how to answer this question. Ask the student to make a table to organize his/her answers from parts a) and b) and see if s/he can make a conjecture about part c).

**Annotation of Student Work With a Score of 1**

- a. 
$$\begin{array}{r} 1.5 \\ + 60 \\ \hline 61.5 \end{array}$$
 
$$\begin{array}{r} 10.5 \\ + 45 \\ \hline 55.5 \end{array}$$
 I got my answer by multi-  
and then comparing
- b. 
$$\boxed{49.3}$$
- c. It will never be a number  
of text where they will be equal because  
they both keep increasing by the same  
amount as before and its either too  
high or too low

**Content Standards:** The student receives a score of 1 because the student responds with appropriate mathematical reasoning in one of the three parts of the problem. Since the two number sentences that the student writes in part a) involve addition, the student's statement, "I got my answer by multiplying," may refer to  $0.05(30) = 1.5$  and  $0.35(30) = 10.5$ ; otherwise, those two numbers are unexplained. The student appropriately adds the 1.5 and 10.5 to the corresponding monthly fee to determine the cost for each plan for 30 texts and circles the lesser amount (8.F.2). No explanation or justification is given in part b). The student's justification in part c) is incorrect.

**Mathematical Practices:** It seems that the student is attempting to make sense of the problem (Practice 1) since s/he provided answers to all three parts, although not completely successfully. In part a) the student demonstrates s/he can reason abstractly and quantitatively (Practice 2). The student is weak on Practice 3, since the statement s/he makes in part a) needs to be extended to justify the calculations that are written and there is no justification given in b). The fact that the student is attempting an explanation in part c) is a positive aspect of the work; however, the student's conclusion is incorrect, "it will never be a number of texts when they will be equal." In addition, the response in part c) is ambiguous, "they both keep increasing by the same amount as before." It is not clear whether the student is thinking correctly that each plan will continue to increase at its given rate of change or if the student is incorrectly thinking both plans increase at the same rate.

**Next Instructional Steps:** Ask the student what s/he is multiplying in part a). Ask the student to give a verbal, then written, explanation for his/her answer in part b). Group the student with other students to discuss and debate various strategies for part c) and challenge the student to revise his/her response.



## GRADE 8 MATH: EXPRESSIONS & EQUATIONS

### INSTRUCTIONAL SUPPORTS

The instructional supports on the following pages include a unit outline with formative assessments and suggested learning activities. Teachers may use this unit outline as it is described, integrate parts of it into a currently existing curriculum unit, or use it as a model or checklist for a currently existing unit on a different topic.

In addition to the unit outline, these instructional materials include:

One sequence of high-level instructional tasks (an arc), including detailed lesson guides for select tasks, which address the identified Common Core Standards for Mathematical Content and Common Core Learning Standards for Mathematical Practice. Each of the lessons includes a high-level instructional task designed to support students in preparation for the final assessment.

Additional high-level instructional tasks, without lesson guides, that can be used to extend the arc or to differentiate the work for particular groups of students

The lessons guides provide teachers with the mathematical goals of the lesson, as well as possible solution paths, errors and misconceptions, and pedagogical moves (e.g., promoting classroom discussion) to elevate the level of engagement and rigor of the tasks themselves. Teachers may choose to use them to support their planning and instruction.

## Unit Outline – 8<sup>th</sup> Grade Math

**INTRODUCTION:** The unit framework consists of an initial formative assessment task, a sequence of lessons that can serve as a formative assessment (an arc), and a final assessment. *Teachers may (a) use this unit as it is described below; (b) integrate parts of it into a currently existing curriculum unit; or (c) use it as a model or checklist for a currently existing unit on a different topic.*

### 8<sup>th</sup> Grade Mathematics: Expressions & Equations

#### UNIT OVERVIEW:

This unit builds directly from prior work on proportional reasoning in 6th and 7th grades, and extends the ideas more formally into the realm of algebra. A description of how these ideas fit together can be found here:

- Progression for the Common Core State Standards in Mathematics (draft) April 22, 2011  
[http://commoncoretools.files.wordpress.com/2011/04/ccss\\_progression\\_ee\\_2011\\_04\\_25.pdf](http://commoncoretools.files.wordpress.com/2011/04/ccss_progression_ee_2011_04_25.pdf)

#### UNIT TOPIC AND LENGTH:

- This sequence of related lessons is intended to take 10-15 instructional days, depending on the length of the class period and students' prior knowledge.

#### COMMON CORE LEARNING STANDARDS:

8.EE Understand the connections between proportional relationships, lines, and linear equations.

- 8.EE.5. Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.
- 8.EE.6. Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for a line intercepting the vertical axis at  $b$ .

8.EE Analyze and solve linear equations and pairs of simultaneous linear equations.

- 8.EE.7. Solve linear equations in one variable.
- 8.EE.7b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.
- 8.EE.8. Analyze and solve pairs of simultaneous linear equations.
- 8.EE.8a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

## Unit Outline – 8<sup>th</sup> Grade Math

- 8.EE.8b. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. For example,  $3x + 2y = 5$  and  $3x + 2y = 6$  have no solution because  $3x + 2y$  cannot simultaneously be 5 and 6.

8.F Define, evaluate, and compare functions.

- 8.F.3. Interpret the equation  $y = mx + b$  as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.

8.F Use functions to model relationships between quantities.

- 8.F.4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
- 8.F.5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. This unit outline highlights the following practices

- MP1. Make sense of problems and persevere in solving them.
- MP3. Construct viable arguments and critique the reasoning of others.
- MP4. Model with mathematics.
- MP6. Attend to precision.

### BIG IDEAS/ENDURING UNDERSTANDINGS:

- Patterns and relationships can be represented graphically, numerically, and symbolically.
- Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportion problems.

### ESSENTIAL QUESTIONS:

- How can patterns, relations, and functions be used as tools to best describe and help explain real-life relationships?
- How can the same mathematical idea be represented in a different way? Why would that be useful?

## Unit Outline – 8<sup>th</sup> Grade Math

### CONTENT:

#### Proportional Relationships

- Unit rate
- Rate of change
- Ratios
- Slope
- Distance/Time
- Similar Triangles
- Table of values
- Scale factor
- Multiplicative versus Additive thinking

#### Linear Equations

- One variable equations
- Slope-intercept form
- Tables, graphs, charts
- Systems of equations
- Real or realistic contexts
- Bivariate data
- Parallel lines

#### Functions

- Input and output
- Independent and dependent variables
- Tables, graphs, charts
- Linear or Nonlinear

### SKILLS:

- **Describe** relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope
- **Find** the unit rate
- **Identify** the rate of change in a table, context, graph and equation
- **Represent** quantities to show if they are proportional
- **Create** an x and y table of values that does not represent a direct proportional relationship

- 
- **Formulate** and **represent** expressions and equations to model a real-world or realistic context in a variety of ways
  - **Describe** the relationship between two quantities in bivariate data (such as arm span and height for students in a classroom) by using linear equations
  - **Solve** linear equations with one variable in a variety of ways
  - **Solve** systems of two linear systems in a variety of ways
  - **Analyze** and **explain** the nature of changes in quantities in linear relationships from a graph
  - **Identify** the similarities and differences of parallel lines
  - **Identify** the equation for linear relationships that are not directly proportional ( $y = mx + b$ )
  - **Represent, analyze, and generalize** a variety of patterns with tables, graphs,

## Unit Outline – 8<sup>th</sup> Grade Math

	<p>words, and, when possible, symbolic rules to make sense of problems</p> <ul style="list-style-type: none"> <li>➤ <b>Write</b> linear functions in the form <math>y = mx + b</math></li> <li>➤ <b>Explain</b> why linear relationships in the form <math>y = mx + b</math> (where <math>b \neq 0</math>) are not proportional</li> </ul> <hr/> <ul style="list-style-type: none"> <li>➤ <b>Recognize</b> that equations for proportions (<math>y/x = m</math> or <math>y = mx</math>) are a special case of the general linear equation (<math>y = mx + b</math>)</li> <li>➤ <b>Identify</b> functions as linear or nonlinear and contrast their properties from tables, graphs, or equations</li> <li>➤ <b>Construct</b> functions to model linear relationships</li> </ul>
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### ASSESSMENT EVIDENCE AND ACTIVITIES:

#### INITIAL ASSESSMENT : WIDGETS

The **initial assessment** also allows for what is sometimes called a *touchstone task*. The task should be rich enough that it can be solved from a variety of approaches, so that students can make sense of it in natural ways. Then as the unit progresses, students should be able to move to more efficient or grade-level appropriate strategies. As the students learn new ideas or procedures, students and the teacher can reflect upon how these new ideas and procedures might apply to the initial task. *See Widgets for assessment details.*

#### FORMATIVE ASSESSMENT:

The purpose of formative assessment is to surface misconceptions and, through the course of the lessons, to provide ways for students to resolve these misconceptions and deepen their understanding. By surfacing misconceptions, the teacher is then able to make mid-unit corrections to instruction. Thus, students' experiences help to improve learning, rather than waiting until the final assessment to uncover problems or gaps in learning. Throughout this unit, periodic collection and analysis of work should yield a wealth of information teachers can use formatively.

#### FINAL PERFORMANCE ASSESSMENT:

At the end of the unit the teacher should give the class the **final assessment** to see how students have improved their thinking and mathematical skills over the course of the instructional unit. This set of tasks assesses students' skills in and knowledge of the big ideas, skills and strategies described in this unit. *See the assessment for full details.*

# Unit Outline – 8<sup>th</sup> Grade Math

## LEARNING PLAN & ACTIVITIES:

The sequence of lessons here (an arc) includes the following tasks:

- Similar Triangles
- Two Oil Tanks
- Creating a Table
- EZ Coasters

They are arranged in a particular order to support the development of the big ideas of the unit. Additional tasks are included in the unit to use as supplementary materials, but do not include detailed lesson guides.

## ADDITIONAL RESOURCES:

### SUPPORTING UNITS FOR STUDENTS

- The following Connected Math Project (CMP) units provide additional investigations and tasks to extend or enrich this unit:
  - *Moving Straight Ahead* (grade 7)
  - *Variables and Patterns* (grade 7)
  - *Say It With Symbols* (grade 8)
  - *Thinking with Mathematical Models* (Grade 8)
  - *The Shape of Algebra* (grade 8)

<http://connectedmath.msu.edu/mathcontent/contents.shtml>

- The following unit from Context for Learning explores the relationship between expressions and equations, all within the larger big idea of equivalence:
  - *The California Frog-Jumping Contest*

<http://www.heinemann.com/products/E01024.aspx>

### ADDITIONAL READING FOR TEACHERS

- Progression for the Common Core State Standards in Mathematics (draft) April 22, 2011  
[http://commoncoretools.files.wordpress.com/2011/04/ccss\\_progression\\_ee\\_2011\\_04\\_25.pdf](http://commoncoretools.files.wordpress.com/2011/04/ccss_progression_ee_2011_04_25.pdf)
- *Young Mathematicians at Work: Constructing Algebra* (Fosnot & Jacob)  
<http://www.heinemann.com/products/E02841.aspx>
- *Fostering Mathematical Thinking: A Guide for Teachers, Grades 6 – 10* (Driscoll)  
<http://www.heinemann.com/products/E00154.aspx>

## SIMILAR TRIANGLES PROBLEM LESSON GUIDE

### LESSON OVERVIEW:

Students will be presented with the Similar Triangles Within a Graph problem. They will be asked to:

- Identify patterns formed within the graph
- Use similar triangles to determine the unit rate
- Create a function table that accurately represents the given data within the graph
- Write an equation ( $y = mx + b$ ) that models the data from the table of values

- **7.RP.2** Recognize and represent proportional relationships between quantities.
  - a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
  - b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
  - c. Represent proportional relationships by equations. *For example, if total cost  $t$  is proportional to the number  $n$  of items purchased at a constant price  $p$ , the relationship between the total cost and the number of items can be expressed as  $t = pn$ .*
    1. Explain what a point  $(x, y)$  on the graph of a proportional relationship means in terms of the situation, with special attention to the points  $(0, 0)$  and  $(1, r)$  where  $r$  is the unit rate.
- **8.EE.5** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.
- **8.EE.6** Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for a line intercepting the vertical axis at  $b$ .

### DRIVING QUESTIONS:

- What patterns exist between the triangles on the graph?
- What is the significance of these patterns? What do they tell us?
- What other ways can we represent the relationship that exists between the triangles?

### NCTM ESSENTIAL UNDERSTANDINGS<sup>1</sup>:

- represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules;
- relate and compare different forms of representation for a relationship; identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations
- understand relations and functions and select, convert flexibly among, and use various representations for them
- explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope

### SKILLS DEVELOPED:

- Find the unit rate
- Use different representations to form ratios and make comparisons with ratios
- Use graphs to analyze the nature of changes in quantities in linear relationships

<sup>1</sup> NCTM (2010) Developing Essential Understandings of Ratios, Proportions & Proportional Reasoning: Grades 6 -8.  
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	<ul style="list-style-type: none"> <li>recognize and generate equivalent forms for simple algebraic expressions and solve linear equations</li> </ul>	
<b>MATERIALS:</b> <ul style="list-style-type: none"> <li>Graph Paper</li> <li>Smartboard</li> <li>Chart paper</li> <li>Colored marker</li> </ul>		<b>GROUPING:</b> <ul style="list-style-type: none"> <li>Independent work/reflection</li> <li>Turn and Talk</li> <li>Groups of 3 to 4 students, selected by the teacher</li> <li>All class introduction followed by groups of 3 to 4 students</li> </ul>

<b>SET-UP: Students will be presented with individual graphs showing the similar triangles</b>	
<b>EXPLORE PHASE: Supporting Students' Exploration of the Mathematical Ideas</b>	
<p><b><u>Private Think Time: 7- 10 minutes</u></b></p> <ul style="list-style-type: none"> <li>Students will be asked to identify and reflect on what they see within the graph and record their findings</li> <li>Students will be asked to find the hourly wage rate of pay</li> <li>Students will be asked to determine the amount of pay for 40 hours of work done</li> </ul> <p><b><u>Small-Group Work: 10 – 15 minutes</u></b></p> <ul style="list-style-type: none"> <li>Students will be asked to turn and discuss their findings with 1 or 2 other classmates</li> <li>Students will be grouped 3 to 4 students and share their different strategies for finding the hourly wage</li> <li>During this part of the process, the teacher will circulate throughout each group and to listen to, record and support different solution paths</li> <li>Depending on the strategies that emerge, a select groups of students will be selected to share their findings to the whole class</li> </ul> <p>As you circulate, identify strategies that you want to be sure to discuss as a whole class during the Share, Discuss, Analyze Phase. Decide on the sequence that you would like these strategies to emerge. Give groups a “heads up” that you will be asking them to come to the front of the room. If a document reader is not available, give selected groups an transparency or chart paper.</p>	
<b>Possible Solution Paths</b>	<b>Possible Assessing and Advancing Questions</b>

<p>Solution Path #1:</p> <p>Graph Interpretation – Student counts 15 units up and 1 unit to the right. This counting is consistently repeated: 15 up/1 right.</p> <ul style="list-style-type: none"> <li>• Student concludes that the hourly rate of pay is \$15 per hour</li> <li>• Student will multiply <math>\\$15 \times 40 = \\$600</math></li> </ul>	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>• In terms of the context, what did each count up mean?</li> <li>• In terms of the context, what did each count to the right mean?</li> <li>• In terms of the context, what does each set of “15 units up and 1 unit to the right” really mean? How do you know?</li> <li>• How did you determine the relationship between the number of hours worked and the amount of pay received?</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>• If nothing changes, will this count continue indefinitely? How could you convince someone in our class who isn’t convinced of this?</li> <li>• Is this idea related to the idea of slope? If so, how?</li> <li>• Would a triangle with a base of 10 and a height of 100 fall along the line of the graph? Explain to your group how you know.</li> </ul>								
<p>Solution Path #2:</p> <p>Looking for a Pattern – <math>15 \div 1 = 15</math></p> <p style="padding-left: 40px;"><math>30 \div 2 = 15</math></p> <p style="padding-left: 40px;"><math>60 \div 4 = 15</math></p> <p style="padding-left: 40px;"><math>105 \div 7 = 15</math></p> <p style="padding-left: 40px;"><math>120 \div 8 = 15</math></p> <ul style="list-style-type: none"> <li>• Student concludes that the hourly rate of pay is \$15 per hour</li> <li>• Student will multiply <math>\\$15 \times 40 = \\$600</math></li> </ul>	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>• I noticed that you used a division strategy. Can you explain why dividing made sense in this problem?</li> <li>• That’s interesting that the quotient continues to be 15. Do you think that will continue to happen? Why or why not?</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>• What relationship is being represented in dividing the amount of pay by the number of hours?</li> <li>• What would happen if you had divided the number of hours by the amount of pay instead? Would this be useful? What would that tell you?</li> <li>• How is the division you did related to the graph or the triangles? Are there any relationships there?</li> </ul>								
<p>Solution Path # 3:</p> <p>Creating a table of values based on the given data</p> <table border="1" data-bbox="138 1328 1129 1487"> <thead> <tr> <th>h. (hours worked)</th><th>w. (wages)</th></tr> </thead> <tbody> <tr> <td>1</td><td>15</td></tr> <tr> <td>2</td><td>30</td></tr> <tr> <td>3</td><td>45</td></tr> </tbody> </table>	h. (hours worked)	w. (wages)	1	15	2	30	3	45	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>• What is the table of values telling you about the number of hours worked and the wages received?</li> <li>• How did you arrive at these values? Was this the only way?</li> <li>• I noticed that as you added 1 to the hours worked column,</li> </ul>
h. (hours worked)	w. (wages)								
1	15								
2	30								
3	45								

4	60	<p>you then added 15 to the wages column. That's a nice strategy, but it will take you awhile. Is there a faster way to get to 40 hours? How can you be sure that it will work?</p> <ul style="list-style-type: none"> <li>How did the pattern that you found in the table help you determine the amount of pay received for 40 hours of work?</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>Is there any connection between this table that you created and the triangles on the graph? What about the line on the graph and your table? Some students are saying they all have something to do with slope. What do you think of this?</li> </ul>
5	75	
6	90	
7	105	
8	120	
40	?	
<p>Pattern</p> <p> <math>1 \times 15 = 15</math>  <math>2 \times 15 = 30</math>  <math>3 \times 15 = 45</math>  <math>4 \times 15 = 60</math>  <math>5 \times 15 = 75</math>  <math>6 \times 15 = 90</math>  <math>7 \times 15 = 105</math>  <math>8 \times 15 = 120</math>  <math>40 \times 15 = 600</math> </p>		
<p>Solution Path #4: Setting up a Proportion</p> <p>_____</p> <p>_____</p> <p>_____</p> <p>_____</p> <p>x = 600 dollars</p>		<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>I see that you used a proportion. That's interesting. What made you convinced that this would work?</li> <li>What do the numbers in your proportion mean in the context of the problem? How do they relate to one another?</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>Can you set up the proportion another way? OR... Another group set up the proportion this way.</li> </ul> <p>_____ What do you think of this strategy?</p> <ul style="list-style-type: none"> <li>How does the ratio 15:1 that you found in part c relate to this new proportion for finding 600 dollars?</li> <li>Look at the graph with the triangles drawn. Is there any connection between the proportional relationship you created and similar triangles? If so, what is it?</li> </ul>

<p>Solution Path #5: Applying a linear equation <math>y = mx + b</math> based on the table of values from solution path #3</p> $y = mx + b$ $15 = \frac{15}{1}x + b$ $15 = 15x + b$ $15 = 15 \cdot 1 + b$ $15 = 15 + 0$ $15 = 15$	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>I see that you created a linear equation. What made you certain that this was a good strategy? In this problem, what might be really helpful about having an equation?</li> <li>If you look at the graph, it's not so hard to determine that the function is linear, but I'm wondering if there's a way to determine this if you only had the table.</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>What does the "15" you found for "m" mean in the context of the problem? Where do you see that in the triangles formed?</li> <li>What does the "0" you found for "b" mean in the context of the problem?</li> <li>How does the ratio of the height to the base of the triangles relate to either the 15 or the 0? Why?</li> <li>Another group of students made a proportion and they are convinced that the data in this problem are proportional. What do you think of this? Is there a way to verify this idea? If you can make a linear equation from some data, does it mean that the data is proportional?</li> </ul>
<p><b>Possible Errors and Misconceptions</b></p> <ul style="list-style-type: none"> <li>Looking at the graph, students may erroneously read the graph and state a 1:1 relationship as opposed to 15:1</li> <li>In using a proportional set-up, students may forget to label and therefore, set up the proportion incorrectly.</li> </ul> <p>Example:                      —     —</p>	<p><b>Possible Questions to Address Errors and Misconceptions</b></p> <p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>We know that graphs tell stories. Tell us what you think this graph is telling us so far. You might look at the x and y axes to start.....</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>Are the x-axis and y-axis both going by the same intervals as shown on the graph?</li> </ul> <p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>What does the 4 represent? The x? The 60 and 40?</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>What if we attached some units to your proportion, just to keep track of what we are comparing? Does it make sense to compare this way?</li> </ul>

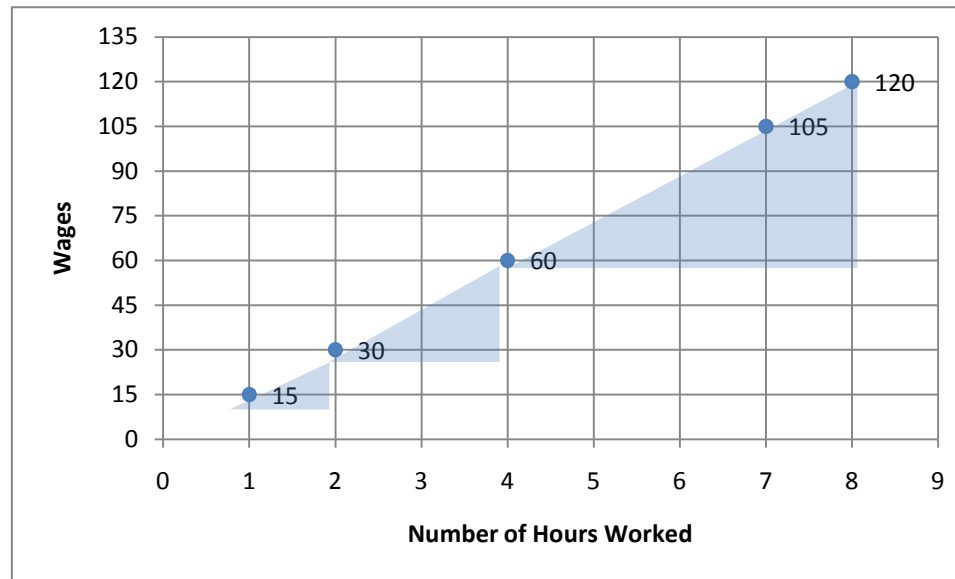
<b>SHARE DISCUSS ANALYZE PHASE: Deepening and Connecting Mathematical Understanding</b>	
<b>General Considerations:</b> <ul style="list-style-type: none"> <li>Faciliate the class discussion so that it builds on, extends, and connects the thinking and reasoning of students</li> <li>Sequence the solution paths so that you will be able to press students to make comparisons and connections across, and between, the various solution paths.</li> </ul>	
<b>Possible Sequence of Solution Paths</b>	<b>Possible Questions and <i>Possible Student Responses</i></b>
Sequence of Solution Paths: <ul style="list-style-type: none"> <li>Solution Path #1: Graphic Interpretation – Student counts 15 units up and 1 unit to the right. This counting is consistently repeated: 15 up/1 right.</li> </ul>	<p><b>Explain your group's solution.</b>  <b>How many of you discovered this pattern within the given graph? How many of you used the exact same strategy?</b></p> <p><b>What does the 15 mean in the context of the problem?</b></p> <p><b>How does the 15 relate to part b, the ratio of the height to the base of the triangles?</b></p> <p><b>What questions do we have for this group?</b></p>
<ul style="list-style-type: none"> <li>Solution Path #2: Looking for a Pattern               <math display="block">15 \div 1 = 15</math> <math display="block">30 \div 2 = 15</math> <math display="block">60 \div 4 = 15</math> <math display="block">105 \div 7 = 15</math> <math display="block">120 \div 8 = 15</math> </li> </ul>	<p><b>Share your group's work with the class.</b>  <b>What was your strategy?</b></p> <p><b>Does your 15 mean the same thing in the context of the problem as the first group's 15? How is that possible?</b></p> <p><b>How does the 15 relate to part c, the slope of the line formed by the data points?</b></p> <p><b>Can anyone name some ways that the first two strategies are related?</b></p>
<ul style="list-style-type: none"> <li>Solution Path # 3: Creating a table of values based on the given data on the graph</li> </ul>	<p><b>How could you use this table to determine the number of hours worked if you were given the total amount of wages received instead?</b></p> <p><b>Does this table also have the SAME 15 as the first two groups did? Does it mean the same thing in the context of the problem?</b></p>
<ul style="list-style-type: none"> <li>Solution Path #4: Setting up a Proportion</li> </ul>	<p><b>Describe your strategy.</b></p> <p><b>You are using the word proportion. Remind us what that means and how you knew that this picture of triangles on a graph was somehow a proportional relationship.</b></p> <p><b>Is it possible to find that 15 AGAIN in a proportion related to this problem? Why or why not?</b></p>

	Are all linear graphs proportional? Turn and talk to your neighbor about whether you think this is true or not. Also, what would convince you of whether this is true or not?
<ul style="list-style-type: none"> <li>Solution Path # 5: Applying a linear equation <math>y = mx + b</math> based on the table of values from solution</li> </ul>	<p>(After making sure that all students are aware of what each variable represents in the context of the problem)</p> <p>Is this STILL the same 15, with the same meaning as all the other groups' 15 meant? Why or why not?</p> <p>Why would you multiply the 15 by <math>x</math>? Why not add it or divide it or do something else to it? What about this problem suggests that it MUST be multiplied by <math>x</math>?</p> <p>Why add 0? Why not multiply it or divide it or do something else to it? What about this problem suggests that 0 MUST be added to the <math>15x</math>?</p> <p>What would it say if some number other than 0 is added to the <math>15x</math>?</p> <p>What equation could you use to determine the number of hours worked if you were given the total amount of wages received instead?</p>

<b>CLOSURE</b>
<p><b>Quickwrite:</b></p> <ul style="list-style-type: none"> <li>Do all graphs that demonstrate proportional relationships share visual characteristics? If so, what are they?</li> <li>What does it mean when we say that there exists a proportional relationship between quantities?</li> <li>What operations (+, -, <math>\times</math>, or <math>\div</math>) are connected to all proportional relationships? Why is this so?</li> </ul>
<p><b>Possible Assessment:</b></p> <ul style="list-style-type: none"> <li>Providing students with a variety of problems, some as word problems, others as graphic representations, and others as tables of values (all within the context of proportional relationships) which various solution paths can be used.</li> </ul>
<p><b>Homework:</b></p> <ul style="list-style-type: none"> <li>Amount of gas used in a travelling car</li> <li>A swimming pool being filled with water at a constant rate</li> </ul>

## Triangles Task

The data shown in the graph below reflects average wages earned by machinists across the nation.



- What hourly rate is indicated by the graph? Explain how you determined your answer.
- What is the ratio of the height to the base of the small, medium and large triangles? What patterns do you observe? What might account for those patterns?

- c. The slope of a line is found by forming the ratio of the change in  $y$  to the change in  $x$  between any two points on the line. What is the slope of the line formed by the data points in the above graph? Explain how you know.
- d. According to the graph, in a 40-hour week, how much will the average machinist earn? How do you know?

## TWO OIL TANKS LESSON GUIDE

### LESSON OVERVIEW:

In this lesson, students will look at two oil tanks that are draining oil at the same rate but with different starting points. Through this task, they will conclude that, when the functions have the same slope but different intercepts, the lines representing them are parallel. They will also note which characteristics of the various representations denote the fact that the lines are parallel.

### COMMON CORE STATE STANDARDS:

- **8.EE.5** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.
- **8.EE.6** Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for a line intercepting the vertical axis at  $b$ .
- **8.F.4** Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
- **8.F.5** Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.
- **7RP.2** Recognize and represent proportional relationships between quantities.  
Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

### NCTM ESSENTIAL UNDERSTANDINGS:

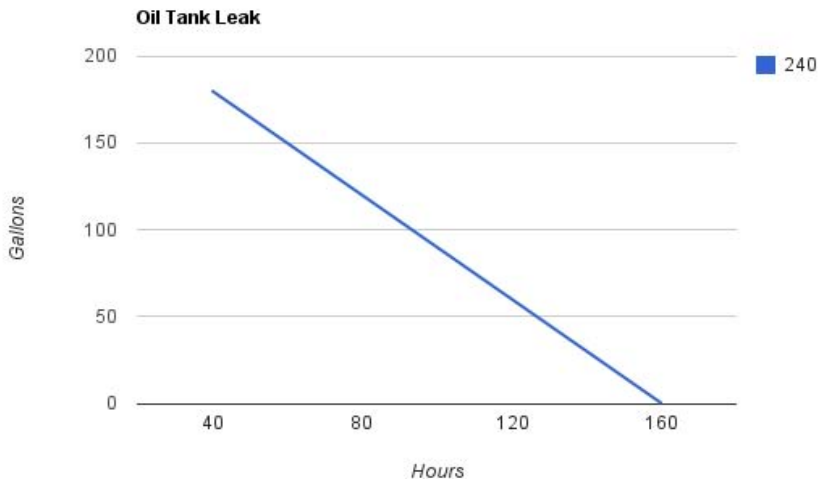
1. Reasoning with ratios involves attending to and coordinating two quantities.
2. A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit.
  - a. Forming a ratio as a measure of a real-world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest.
3. A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.
4. Proportional reasoning is complex and involves understanding that:
  - a. Equivalent ratios can be created by iterating and/or partitioning a composed unit;
  - b. If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and
  - c. The two types of ratios – composed units and multiplicative comparisons – are related.
5. A rate is a set of infinitely many equivalent ratios.
6. A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.
7. Proportional reasoning is complex and involves understanding that:
  - a. Equivalent ratios can be created by iterating and/or partitioning a composed unit;
  - b. If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to

<p>maintain the proportional relationship; and</p> <p>c. The two types of ratios – composed units and multiplicative comparisons – are related.</p> <p>8. A rate is a set of infinitely many equivalent ratios.</p>		
<p><b>DRIVING QUESTION:</b></p> <ul style="list-style-type: none"> <li>By using a variety of strategies (and representations) how can we determine if two quantities have a proportional relationship?</li> <li>When two things share the same rate of change, what is similar about their tables, graphs and equations?</li> <li>When two things share the same rate of change, what might be different about their tables, graphs and equations?</li> </ul>	<p><b>NCTM ESSENTIAL UNDERSTANDINGS<sup>2</sup>:</b></p> <p>4. Proportional reasoning is complex and involves understanding that:</p> <p>b. If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and</p> <p>5. A rate is a set of infinitely many equivalent ratios.</p>	<p><b>Skills developed:</b></p> <p>Students will be able to:</p> <ul style="list-style-type: none"> <li>Determine whether or not two quantities are in proportional relationship using a variety of strategies (and representations)</li> <li>Write linear functions in the form <math>y=mx + b</math></li> <li>Identify the rate of change in a table, context, graph and equation</li> <li>Determine that lines with the same slope and different intercepts are parallel</li> </ul>
<p><b>SET-UP</b></p>		
<p><b>Instructions to Students:</b></p> <p>The Oil Tank Task will be presented to the class via a document reader or overhead projector. A student can read the task and the associated prompts.</p> <p><b>Help students to begin problem formulation:</b> After listening to the task, what do you know? What is this problem about? And what is it asking you to do? [Students can think about this for a few moments and then discuss as a class, or do some independent writing and then share with a partner.]</p> <p>Teacher informs class that there may be several ways to find the answers to the given questions. Every student will be given 10 minutes of alone time to begin working on the task. Then students will work in threes/fours to compare their findings and to continue to work on a polished presentation of their strategy and solution.</p> <p><b>Class expectations:</b> All students should be encouraged and supported to justify their thinking and mathematical reasoning, use precise mathematic language and symbols, and make sense of the solutions offered by other students. Students will seek help from the teacher only after they have sought help from each other and still do not understand.</p>		
<p><b>EXPLORE PHASE: Supporting Students' Exploration of the Mathematical Ideas</b></p>		
<p><b>Private Think Time:</b> Allow students to work individually for 3-5 minutes without intervening, though you should circulate quickly to get an idea of the strategies that are emerging in the classroom.</p> <p><b>Small-Group Work:</b> After 3-5 minutes, ask students to work with their partner or in their small groups. As students are working, circulate around the room. Be persistent in:</p> <ul style="list-style-type: none"> <li>asking questions related to the mathematical ideas, problem-solving strategies, and connections between representations</li> </ul>		

<sup>2</sup> NCTM (2010) Developing Essential Understandings of Ratios, Proportions & Proportional Reasoning: Grades 6 -8.  
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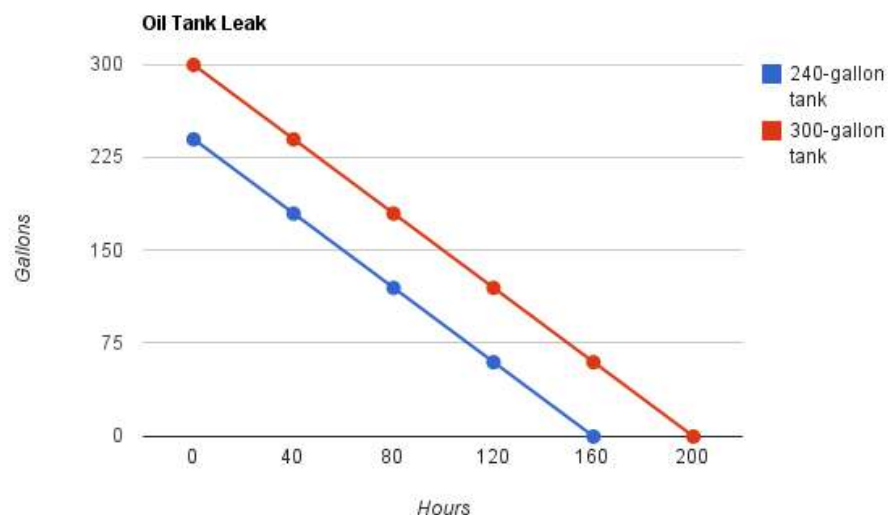
- asking students to explain their thinking and reasoning, including interpreting the meaning of their strategies and various parts of their solution
- asking students to explain in their own words, and build onto, what other students have said

As you circulate, identify solution paths that you want to be sure to discuss together during the Share, Discuss, Analyze phase. Decide on the sequence that you shows the progression of mathematical ideas. Give groups a “heads up” that you may be asking them to come to the front of the room. If a document reader is not available, give selected groups a transparency or chart paper to record their solution.

Possible Solution Paths	Possible Assessing and Advancing Questions
<p>If a group is unable to start:</p> <ul style="list-style-type: none"> <li>• <i>Well, it's draining, so <math>200 - 1.5</math> or 238.5 gallons. <math>240 - 3</math> or 270 gallons. <math>240 - 15 = 225</math>.</i></li> </ul>	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>• Let's think about Tank A. How many gallons are in the tank? How many will be left after an hour? Two? This is going to take awhile, huh? What about using a larger value, instead of one hour each time, to drain the tank? What would be a good number to choose? Why might that help us?</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>• Okay, given all of this data, how can we organize it better in order to draw the graph? Do we need all of it, or some of it? What do you think?</li> </ul>
<p>a. Create a graph and write the equation: <math>-1.5x + 240 = y</math></p>  <p>The relationship is not proportional because it does not pass through the origin (0, 0). The</p>	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>• How did you use the graph to determine the equation of the line?</li> <li>• Is there a way to “see” the -1.5 in this graph? First, what does the -1.5 mean in the context of the oil tank?</li> <li>• What about the 240? Where did that number come from? And what does it mean in the context of the oil tank?</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>• Is there a proportional relationship between the number of hours and the number of gallons? How do you know?</li> <li>• Predict how the graph of the tank filled with 300 gallons will look like. What similarities and differences between the situations can you use to draw that graph?</li> </ul>

x values are not constant multiples of the y values (or the y values are not constant multiples of the x values.)

b. **Graph both lines but only write the equation:**  $-1.5x + 240 = y$



The two lines are parallel, they have the same rate of change, but they start at different places.

### Assessing Questions

- What process did you use to graph the lines? What's the same about these graphs? What's different? Why does that make sense given the story of the oil tanks?
- Is there a way to "see" the  $-1.5$  in this graph? First, what does the  $-1.5$  mean in the context of the oil tank?
- What about the 240? Where did that number come from? And what does it mean in the context of the oil tank?
- How about the 300? Where did that number come from? And what does it mean in the context of the oil tank?

### Advancing Questions

- Thinking about the strategy that you used and looking at the graph, what conclusions can you draw about the slope and y-intercept of both lines?
- How can you use those similarities/differences to find the equation of the line for the each tank?
- Will there ever be a time when both lines will meet? How do you know? Convince us.

c. **Graph both lines (see above), write both equations:**  
 $-1.5x + 240 = y$  and  $-1.5x + 300 = y$

### Assessing Questions

- What strategy did you use to graph the lines? And so, what is the same about these graphs? And what's not? Why do you think that is?
- What similarities and differences did you notice in the equations? Why?

### Advancing Questions

- Will there ever be a time when both lines will meet? How do you know?

	<ul style="list-style-type: none"><li>When will each tank become empty? How do you know?</li><li>If we think about a third tank, maybe 200 gallons full, predict what the graph and equation will look like. What would you say to another student in our class to convince him/her of your prediction?</li></ul>																					
<p><b>d. Find an equation:</b></p> <p>The equation is <math>-1.5x + 240 = y</math>. The <math>-1.5</math> is the rate of change for both situations. Since the tank is losing gallons, that is why the rate is negative. Since it is losing 1.5 gallons an hour, that is where the 1.5 comes from. The tank starts at 240 gallons so that is the y-intercept.</p>	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"><li>How did you determine the equation of this line?</li><li>What did you say the slope of that line is and what does it mean in terms of the oil tanks?</li></ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"><li>Does the equation of the line represent a proportional relationship? How do you know? Convince your group and then think about how you might convince the whole class.</li><li>Predict what the equation of the tank filled with 300 gallons will look like. What similarities and differences between the situations can you use to write that equation?</li><li>What do you expect the graphs will look like? Why? What is it about these two situations that helps you to be convinced of your idea?</li></ul>																					
<p><b>e. Make a Table</b></p> <table><tr><th>Hours</th><th>240 gallon Oil Tank</th><th>300 Gallon Oil Tank</th></tr><tr><td>0</td><td>240</td><td>300</td></tr><tr><td>40</td><td>180</td><td>240</td></tr><tr><td>80</td><td>120</td><td>180</td></tr><tr><td>120</td><td>60</td><td>120</td></tr><tr><td>160</td><td>0</td><td>60</td></tr><tr><td>200</td><td></td><td>0</td></tr></table>	Hours	240 gallon Oil Tank	300 Gallon Oil Tank	0	240	300	40	180	240	80	120	180	120	60	120	160	0	60	200		0	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"><li>How did you determine the numbers to put in your table?</li><li>Say more about these numbers. Why is one column increasing, while the others are decreasing. Why do both oil tank columns end at zero? Why did you stop there? How come they don't reach zero at the same time? Are they draining at the same rate? What evidence from the table tells you this?</li></ul>
Hours	240 gallon Oil Tank	300 Gallon Oil Tank																				
0	240	300																				
40	180	240																				
80	120	180																				
120	60	120																				
160	0	60																				
200		0																				

	<b>Advancing Questions</b> <ul style="list-style-type: none"> <li>• How can you use the data in your table to create a graph?</li> <li>• What do we mean by the slope of a line? How can you use the table to determine the slope of the line?</li> <li>• What do you notice about the numbers used to determine the slopes of both lines? So what does this tell you?</li> </ul>
<b>Possible Errors and Misconceptions</b>	<b>Possible Questions to Address Errors and Misconceptions</b>
<p>Table Errors: Students may use inconsistent intervals in their table, and/or only find the difference in the y-values to determine slope.</p> <p>Graphical Errors:</p> <p>(a) Students could have inconsistent intervals on the scale of their graphs, possibly from using inconsistent intervals in the tables or sets of points they have created.</p> <p>(b) students could reverse the points and plot them incorrectly</p>	<b>Assessing Questions</b> <ul style="list-style-type: none"> <li>• What do we mean by slope? Have you used that in your work?</li> </ul> <b>Advancing Questions</b> <ul style="list-style-type: none"> <li>• What do you notice about the ratios you are now forming? Why is that happening?</li> </ul> <b>Assessing Questions</b> <ul style="list-style-type: none"> <li>• (Pointing to one interval) How large is this interval? (Then pointing to another interval) And this one? What could this mean? OR (if a graph exists on the axes)</li> <li>• You expected that this data would form a line because of the constant rate of change. What might be causing your graph to look not so much like a line?</li> </ul> <b>Advancing Questions</b> <ul style="list-style-type: none"> <li>• Will the change in scale matter? Why?</li> </ul> <b>Assessing Questions</b> <ul style="list-style-type: none"> <li>• (Pointing to one point) What does the x-value of this point mean in terms of the oil tanks? And the y-value?</li> </ul> <b>Advancing Questions</b> <ul style="list-style-type: none"> <li>• In this situation, which variable, x or y, is the independent variable? The dependent variable?</li> </ul>

<p>Equation Error – students could write incorrect equations, e.g., <math>240x - 1.5 = y</math></p>	<p><b>Assessing Questions</b></p> <ul style="list-style-type: none"> <li>(Pointing to the 240 and the x) What does the x-value represent in your equation? What will you find if you multiply 240 times the time, according to the problem?</li> </ul> <p><b>Advancing Questions</b></p> <ul style="list-style-type: none"> <li>Since you have said x represents time, what did you do here (pointing to a point on the graph or in the table) to find the remaining gallons? How can you use that information to find the equation of the line in this situation?</li> </ul>
<p><b>SHARE DISCUSS ANALYZE PHASE: Deepening and Connecting Mathematical Understanding</b></p>	
<p><b>General Considerations:</b></p> <ul style="list-style-type: none"> <li>Orchestrate the class discussion so that it builds on, extends, and connects the thinking and reasoning of students</li> <li>Sequence the solution paths so that you will be able to press students to make comparisons and connections across, and between, the various solution paths.</li> </ul>	
Possible Sequence of Solution Paths	Possible Questions and <i>Possible Student Responses</i>
<p>1. <b>Start with the students who began with a table of values</b></p>	<p><b>Explain your group's solution.</b></p> <ul style="list-style-type: none"> <li>Although the first question required us to draw a graph, we began by making a table, so that we could get the correct values to plot. Because the rate of drainage was 1.5 we chose large number values for the hours, multiplied by 1.5 and subtracted from 240 (or 300). It would take too long to plot if we used smaller numbers.</li> <li><b>How can you use the data in your table to determine whether or not the relationship is proportional?</b> We could select any two points and figure out whether the x and y values are the same multiples of one another, or make ratios and see if they are equivalent, or we could look at the value for y when x is 0. If this value is not 0 then we could conclude that it is not proportional.</li> </ul>
<p>2. <b>Have students share their graphical solution</b></p>	<ul style="list-style-type: none"> <li><b>How did you choose the values for the graph?</b> We selected large consistent numbers for the hours so that we could create the graph easily. Also because the start number was large, we decided that large numbers would be better so that we could empty the tank quickly. Then we said, draining at 1.5 gal./hr. means down 1.5 for every right 1. Well, that's like, down 15 for every right 10. We counted off</li> </ul>

	<p>the slope in steps and formed our lines.</p> <ul style="list-style-type: none"> <li>• <b>How many people can explain in their own words, why 1.5 for 1 is the same as 15 for 10? Turn and talk to your partner about how this group knew this and how they used it to help them.</b></li> <li>• <b>How can you tell whether the slope of the line on the graph is positive or negative? Does the context of the oil tanks help us with that?</b> One way is to look at the x and y values. In a negative slope situation, as the x values increase the y values decrease. Also the rise goes down even though the run goes to the right. And anyway, the tank is draining, so it makes sense that the gallons would be decreasing as the time increases.</li> <li>• <b>Will the values from the first group's table appear on your graph? Why or why not?</b> Yes, all the values on our line can be found by multiplying time by -1.5 and then adding 240 (or 300). Since that's how they found their points, then their points are on our lines.</li> <li>• <b>Is it possible to see the "down 1.5 for every right 1 or down 15 for every right 10" in the table, then? Turn and talk to your partner for one minute, then we'll have someone not from this group explain to all of us.</b> Sure. Look - the tank numbers both keep going down by 60s in the table, but the time goes up by 40s. <math>-60/40 = -15/10 = -1.5/1</math></li> <li>• <b>How can you tell from the graph whether or not the relationship is proportional? Is there more than one way to verify this?</b> Same as before...We could select any two points and figure out whether the x and y values are the same multiples of one another, or make ratios and see if they are equivalent, or we could look at the origin. If the graph does not pass through the origin, we conclude that it is not proportional.</li> </ul>
<p><b>3. Have students share their equation</b></p>	<ul style="list-style-type: none"> <li>• <b>Explain how you derived the two equations.</b> In order to create the equations, we used the information from the question. We used the rate of drainage 1.5 to create an algebraic expression with a negative slope, since the oil tank was losing oil. So we wrote <math>-1.5x</math> and then we added the quantity of oil that each tank had when they began. So the equations looked like this. <math>y = 1.5x + 240</math> and <math>y = -1.5x + 300</math>. It was very easy to write.</li> <li>• <b>How can you tell from the equation whether or not the relationship is proportional?</b> We know that, for the line to be proportional, the y-value is always</li> </ul>

	<p>a multiple of the rate of change, with no added value. In this problem both equations have an added value, 240 and 300. So y is not a multiple of x. Not proportional.</p>
	<ul style="list-style-type: none"> <li> <b>Look at the graph again. These lines appear to be parallel. Did any group confirm that they are?</b>            Yes, we did – TWO ways. First, we COUNTED the difference between any two points on the graph. They were always the same. (Teacher invites student to come and show this on the paper). Then, we played with the numbers. Look at that table again...can you see how the y-values are always 60 apart? Well, we said, “Hey, that’s the definition of parallel lines. Equally distant apart.”         </li> <li> <b>Wow! So what might the equations tell us?</b>            Same thing! Look...<math>(-1.5x + 300) - (-1.5x + 240) = 60</math>.         </li> <li> <b>But why did you subtract as you did?</b>            The y-value of the top line minus the y-value of the bottom line is always 60. When you subtract y-values, you get the distance between the lines. Equally distant apart.         </li> <li> <b>Okay. So how can we be certain that two things are really going to be parallel when graphed? Convince us of this using the equations, or the tables, or even the context itself.</b> </li> </ul>

<b>CLOSURE</b>
<ul style="list-style-type: none"> <li>• <b>Quickwrite:</b> How can use an equation, table, graph or context to tell if the lines on the graph will be parallel?</li> </ul>
<b>Possible Assessment:</b> <ul style="list-style-type: none"> <li>• Give students a similar problem and have them use mathematical reasoning to justify their answers.</li> </ul>
<b>Homework:</b> <ul style="list-style-type: none"> <li>• Provide students two or three different types of problems in order to assess their proportional reasoning abilities.</li> </ul>

## Two Oil Tanks

A full 240-gallon oil tank is being drained at the rate of 1.5 gallons / hour.

- a. Graph time in hours vs. gallons left in the tank and determine whether the relationship is proportional. Explain how you made your decision.
- b. Find the equation of the line and explain the meaning of the slope in the context of the problem.
- c. On the same axes, graph the line for a 300-gallon tank being drained at the rate of 1.5 gallons/hour.
- d. Compare the graphs and equations of the two lines and explain any similarities and differences you observe.

## CREATING A TABLE EIGHTH GRADE LESSON GUIDE

### LESSON OVERVIEW:

#### Task:

The students are to create two x and y tables of values, one representing a proportional relationship and one that does not represent a linear, non-proportional relationship.

#### Goals:

- The students will create x and y tables of values to demonstrate their understanding of proportional and non-proportional linear relationships.
- The students will write the equation that models ( $y = mx$ ;  $y = mx+b$ ) the table they created.
- The students will discuss, explain and write about how can they test and recognize proportional and non-proportional linear relationships, from a table, an equation, a graph and by multiplying or dividing values of x and y by some constant.

### COMMON CORE STATE STANDARDS:

- **8.EE.5** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.
- **8.EE.6** Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for a line intercepting the vertical axis at  $b$ .

#### DRIVING QUESTIONS:

1. If all linear functions represent a steady rate between the x and y values, aren't they then all proportional? Why or why not?
2. What do we need to know about linear relationships to be able to create two tables of values, one that represent proportion and one that does not?
3. How can we tell if the relationship, between the variables x and y is proportional by looking at a table of values, an equation, a graph or in words?
4. What evidence could we use to convince ourselves that the linear relationship between x and

#### NCTM ESSENTIAL UNDERSTANDINGS<sup>3</sup>:

1. A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains the constant as the corresponding values of the quantities change.
2. A crucial property, of linear relationships, involves a constant rate of change.
3. The rate of change is the amount that y changes when x changes by 1 unit.
4. A relationship between x and y is linear if  $y = mx+b$ , where m and b are constants. Here the constant **m** describes the rate of change and the constant **b** describes y-intercept.
5. When the constant term  $b = 0$ , the quantity y is proportional to x. This proportional linear relationship is represented by the equation  $y = mx$ .

#### SKILLS DEVELOPED:

1. Create an x and y table of values that represents proportional relationships
2. Create an x and y table of values that does not represent a direct proportional relationship.
3. Multiply or divide the x and y values by a constant to test for proportionality
4. Identify the general equation for linear relationships that are directly proportional ( $y = mx$ ), and explain why this is the case
5. Identify the equation for linear relationships that are not directly proportional ( $y = mx+b$ ),

<sup>3</sup> NCTM (2010) Developing Essential Understandings of Ratios, Proportions & Proportional Reasoning: Grades 6 -8.  
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y is directly proportional?		and explain why this is the case 6. Communicate orally and in writing their understanding of directly proportional and not directly proportional linear relationships
<b>MATERIALS:</b> <ul style="list-style-type: none"> <li>• 11" by 17" size paper</li> <li>• Rulers</li> <li>• Activity sheets</li> </ul>		<b>GROUPING:</b> Assign students a partner or small group that will both support and push their thinking. While students may do some independent thinking about the task before they begin, they are strongly encouraged to work in collaboration with their peers.

SET-UP
<p><b>Instructions to Students:</b></p> <p><b>Aim:</b> How do I create a table of <b>x</b> and <b>y</b> tables of values, one showing a proportional relationship and one showing a non-proportional relationship?</p> <p><b>Task:</b>  Create a table of x and y values that represents a proportional relationship.  a) Explain how you know the relationship is proportional.  b) What equation models the values in the table?</p> <p>Create a table of x and y values that represents a non-proportional linear relationship.  a) Explain how you know the relationship is non-proportional.  b) What equation models the values in the table?</p> <p><b>Steps:</b>  With a partner or small group that I have assigned:  1. Use the paper provided to create two <b>x</b> and <b>y</b> tables: one proportional, one non-proportional.  2. Make sure your work is clear and visible if I posted it at the front of the room.  3. Answer the questions below together and explain in writing: <ul style="list-style-type: none"> <li>• When looking, just at the equation, how can you tell if the relationship, between the variables x and y is proportional or non-proportional?</li> <li>• What methods can you use to prove it?</li> </ul> </p>

## EXPLORE PHASE: Supporting Students' Exploration of the Mathematical Ideas

**Private Think Time:** Allow students to work individually for 3-5 minutes without intervening, though you will want to circulate quickly to get an idea of the strategies that they are using.

**Small-Group Work:** After 3-5 minutes, ask students to work with their partner or in their small groups. As students are working, circulate around the room. Be persistent in:

- asking questions related to the mathematical ideas, problem-solving strategies, and connections between strategies.
- asking students to explain their thinking and reasoning.
- asking students to explain in their own words, and extend what other students have said.

As you circulate, identify solution paths that you will have groups share during the Share, Discuss, Analyze Phase, and decide on the sequence that will support the progression of mathematical ideas. Give groups a “heads up” that you may be asking them to come to the front of the room. If a document reader is not available, give selected groups a transparency or chart paper to record their solution.

### Possible Solution Paths

- Create an x and y table of values that represents proportional relationships.
- Write the x values from 0 to at least 5.
- Decide on a constant rate of change. When the x value changes by 1, by how much will the value of y change?
- What is the value of y when  $x = 0$ ; when  $x = 1$ ; when  $x = 2$ , and so on?
- Test for proportion. Double the value of x. Did the value of y also double?
- What if you triple it?
- Write the equation that models the values in the table.
- Write the equation for your table.
- Create an x and y table of values that does not represent proportional relationships.
- Write the x values from 0 to at least 5.
- Decide on a constant rate of change. When the x value changes by 1, by how much will the value of y change?
- What is the value of y when  $x = 0$ ; when  $x = 1$ ; when  $x = 2$ , and so on?
- Test for proportion. Double the value of x. Did the value of y also double?
- What if you triple it?

### Possible Assessing and Advancing Questions

#### Assessing Questions

- So, what's this task about? What are you being asked to do?
- So how could you start?
- If the relationship between x and y is directly proportional, what do we know for sure? What has to be true?
- Have you decided on a constant rate of change? For example, have you decided by how much the value of y will change when x changes by 1?
- If the relationship between x and y is non-proportional, what do we know for sure? What has to be true here?
- How can you be sure that this is a proportional relationship?
- How can you be sure that this is a non-proportional relationship?
- How could you convince someone who isn't so sure?

#### Advancing Questions:

- If I ask you to write a linear equation that models proportional relationships, what would you write?
- If I ask you to write a linear equation that models relationships between x and y, that are non-proportional, what would you write?



	<ul style="list-style-type: none"> <li>On the Verizon and AT&amp;T plans, what determined the cost? So, every time you added a minute the cost increased by what _____?</li> <li>Would that be the constant rate of change? Why or why not?</li> </ul> <p>Advancing Questions</p> <ul style="list-style-type: none"> <li>Where would you find the rate of change in the equation?</li> <li>When you graph the line of an equation, how does the line relate to the rate of change?</li> <li>What can you tell about the rate of change of two lines on a graph?</li> <li>How does the steepness of the line compare to each rate of change?</li> <li>On a graph, the lines for the equations: <math>y = 4x</math>, <math>y = 4x + 6</math> and <math>y = 4x - 2</math>, are parallel. Why do you think that is? What make these lines parallel?</li> <li>Were the lines for the equations representing the cost per minute for the AT&amp;T and Verizon plans, parallel? Why was that?</li> </ul>
<b>SHARE DISCUSS ANALYZE PHASE: Deepening and Connecting Mathematical Understanding</b>	
<p><b>General Considerations:</b></p> <ul style="list-style-type: none"> <li>Plan for [make a list the questions you will ask in advance] and facilitate the class discussion so that it builds on, extends, and connects the thinking and reasoning of students</li> <li>Sequence the solution paths so that you will be able to support students to make comparisons and connections across, and between, the various strategies.</li> </ul>	
<b>Possible Sequence of Solution Paths</b>	<b>Possible Questions and <i>Possible Student Responses</i></b>
<ul style="list-style-type: none"> <li>The students will place their work on the board under the columns proportional relationship and not a proportional relationship.</li> <li>The students will explain their work and why their work was placed on the proportional and not proportional relationship column.</li> <li>The students will make generalizations about proportional and not proportional relationships.</li> <li>The students will determine if the relationship is directly proportional by just looking at a table of values,</li> </ul>	<ul style="list-style-type: none"> <li>Let's look at the work on the board.</li> <li>What do you see? What jumps out at you?</li> <li>Do you think they are all in the right place? Are there any that might be in the wrong place? What tells you this? Does any group want to change their poster, or relocate it to a different place on the board?</li> <li>I'm going to ask a few of you to explain your work and how you knew where to place it on the board.</li> </ul>

equations, or graphs representing linear relationships.	
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The students' work will be posted on the board as shown below:

	Proportional Relationship	Non-Proportional Relationship
Table of Values		
Equation		
Graph		

- What do you observe from the table of values, equations and graphs under each of the columns?
- What generalizations can you make about proportional linear relationships from the table of values, equations and graphs, under each column?
- What generalizations can be made about an x and y table of values representing linear relationships that are proportional?
- What generalizations can be made about an x and y table of values representing linear relationships that are not directly proportional?
- What generalizations can be made about an equation representing linear relationships that are proportional?
- What generalizations can be made about an equation representing linear relationships that are non-proportional?
- What generalizations can be made about a graph representing linear relationships that are proportional?
- What generalizations can be made about a graph representing linear relationships that are non-proportional?

## CLOSURE

### Quick Write:

- How can you tell if a linear relationship between the variables x and y is proportional or non-proportional by looking only at the table of values?
- How can you tell if a linear relationship between the variables x and y is proportional or non-proportional by looking only at the equation?
- How can you tell if a relationship between the variables x and y is proportional or non-proportional by looking only at the graph?

**Possible Assessment:**

Name \_\_\_\_\_

Date \_\_\_\_\_

Class \_\_\_\_\_

**Quiz**

Analyze the two tables below. Identify and explain which table represents a linear relationship that is proportional and the one that it is non-proportional.

# 1 Table of Values

x	0	1	2	3	4	5
y	0	4	8	12	16	20

#2 Table of Values

X	0	1	2	3	4	5
y	5	7	9	11	13	15

Demonstrate your understanding of proportional and non-proportional linear relationships by completing the chart below:

Task	A Proportional Relationship	Non- Proportional Relationship
<ul style="list-style-type: none"><li>Write the table of values number (#1 and #2) under the column that represents the relationship between x and y.</li><li>Explain how you know.</li></ul>		
<ul style="list-style-type: none"><li>Find the equation for each table and place it under the correct column.</li><li>How did you know where to place each equation?</li></ul>		
<ul style="list-style-type: none"><li>Make a sketch of what the equations would</li></ul>		

each look like if you graphed them in the first quadrant.		
<ul style="list-style-type: none"> <li>Under the appropriate column, write three ways you can tell if the relationship between the variables <math>x</math> and <math>y</math> is proportional or non-proportional.</li> </ul>		

## CREATING A TABLE TASK

1. Create a table of  $x$  and  $y$  values that represents a proportional relationship.
  - a) Explain how you know the relationship is proportional.
  - b) What equation models the values in the table?
2. Create a table of  $x$  and  $y$  values that represents a linear, non-proportional relationship.
  - b) Explain how you know the relationship is non-proportional.

c) What equation models the values in the table?

## QUICK WRITE

How can I determine if the linear relationship between the variables  $x$  and  $y$  is proportional?

How can I determine if the linear relationship between the variables  $x$  and  $y$  is directly proportional?

## EZ COASTERS LESSON GUIDE

In the *EZ Coasters* task, students encounter an open-ended problem in which they are asked to compare the cost of buying beverage coasters from two companies. Students will likely approach the task using a range of different strategies, making comparisons between the companies with equations, graphs and tables. In some instances, the cost of buying the coasters is a proportional relationship, while in others, it is not. The intent of the task is to encourage students to move among representations, and recognize linear relationships in all three representations. In addition, students will examine a graph and use slope and intercept to determine which coaster relationship is described there. Students will begin to recognize that some of the lines will intersect. Students should have experiences with linear situations in all representations previous to this lesson.

The strategies for comparing the representations will be compared and connected during the whole-group discussion. How to determine and connect slope,  $m$ , and intercept,  $b$ , in each situation will be examined. Students should be able to see how each form, table, graph and equation, provides information needed to derive one of the other forms.

### COMMON CORE STATE STANDARDS:

- **7.RP.2** Recognize and represent proportional relationships between quantities.
  - d. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
  - c. Represent proportional relationships by equations. *For example, if total cost  $t$  is proportional to the number  $n$  of items purchased at a constant price  $p$ , the relationship between the total cost and the number of items can be expressed as  $t = pn$ .*
- **8.EE.5** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.
- **8.EE.6** Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for a line intercepting the vertical axis at  $b$ .
- **8.F.1** Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.
- **8.F.4** Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values

### DRIVING QUESTION:

- How does analyzing data for linearity and proportionality support finding the equation of a line?

### NCTM ESSENTIAL UNDERSTANDINGS<sup>4</sup>:

- 6. A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.
- 7. Proportional reasoning is complex and involves understanding that:
  - a. Equivalent ratios can be created by iterating and/or partitioning a

### SKILLS DEVELOPED:

Students will be able to make connections between data presented in assorted representations. They will be able to interpret slope and intercept within the context of a

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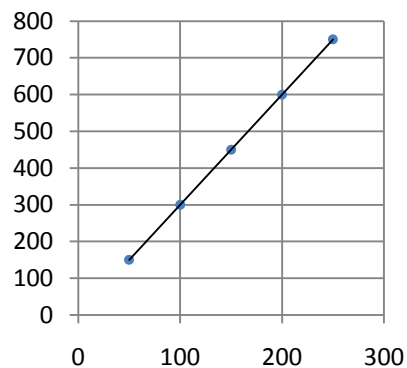
<ul style="list-style-type: none"> <li>Where do we “see” slope in a table, graph, or equation? What does slope mean in the context of this problem and others like it?</li> </ul>	<ul style="list-style-type: none"> <li>composed unit;</li> <li>If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and</li> <li>The two types of ratios – composed units and multiplicative comparisons – are related.</li> </ul>	<p>problem.</p>
<p>8. A rate is a set of infinitely many equivalent ratios.</p> <p>9. Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportion problems.</p>		
<p><b>MATERIALS:</b>  <i>EZ Coasters</i> task sheet, calculators, graph paper. Either a document reader, overhead transparencies, or chart paper.  Beverage coasters of assorted types to clarify what they are for ELL and other students who may only be familiar with the use of the word “coaster” in a roller-coaster context.</p>	<p><b>GROUPING:</b>  Students will begin their work individually, but will then work in groups of three or four to discuss the task and arrive at a common solution.</p>	

<p><b>SET-UP</b></p> <p><b>Instructions to Students:</b></p> <p>Ask a student to read the task while others follow along. Ask students: “What do we know?” “What do we want to find out?” Tell students that there are many ways to reason about the questions. The phrase “set-up fee” may need explaining. First ask for students who understand what the term means to explain, and if not, explain yourself.</p> <p>Remind students that, as in previous work together, they will be expected to:</p> <ul style="list-style-type: none"> <li>justify their solutions;</li> <li>explain their thinking and reasoning to others;</li> <li>make sense of other students’ explanations;</li> <li>ask questions of the teacher or other students when they do not understand;</li> <li>and use precise mathematical language and symbols.</li> </ul>
<p><b>EXPLORE PHASE: Supporting Students’ Exploration of the Mathematical Ideas</b></p> <p><b>Private Think Time:</b> Allow students to work individually for 3 – 5 minutes without intervening, though you might want to circulate quickly to get an idea of the strategies that they are using.</p> <p><b>Small-Group Work:</b> After 3 – 5 minutes, ask students to work with their partner or in their small groups. As students are working, circulate around the room. Be persistent in:</p> <ul style="list-style-type: none"> <li>asking questions related to the mathematical ideas, problem-solving strategies, and connections between representations.</li> <li>asking students to explain their thinking and reasoning.</li> <li>asking students to explain in their own words, and extend what other students have said.</li> </ul> <p>As you circulate, identify solution paths that you will discuss during the Share, Discuss, Analyze phase, and decide on the sequence that you would like</p>

for them to be shared. Give groups a “heads up” that you may be asking them to come to the front of the room. If a document reader is not available, give selected groups a transparency or chart paper record their solutions.

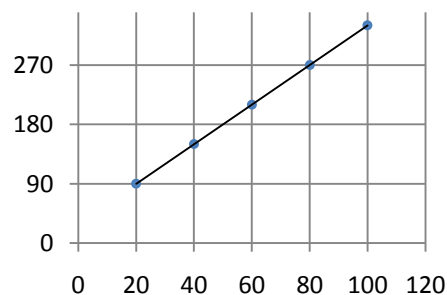
Possible Solution Paths	Possible Assessing and Advancing Questions
<p>If a group is unable to start Example 1, ask what patterns they observe in the values created by ascending numbers of coasters.</p> <ul style="list-style-type: none"> <li>• \$3, \$6, \$9. <i>We multiply the number of coasters by 3.</i></li> <li>• <i>For the coasters with a logo, after we multiply by 3, we add \$25.</i></li> </ul>	<p>Assessing Questions</p> <ul style="list-style-type: none"> <li>• <i>What are some strategies for examining this type of problem that we have used in the past?</i></li> <li>• <i>Tell us about the cost of one plain coaster (then two or three, as necessary.) What patterns are you noticing?</i></li> </ul> <p>Advancing Questions</p> <ul style="list-style-type: none"> <li>• <i>What if we had <math>n</math> coasters? What will you do if the coasters are plain coasters?</i></li> <li>• <i>How will that change if they are coasters with a logo?</i></li> <li>• <i>Can you make a table or a graph to help you think about this problem?</i></li> </ul>
<p>If a group is unable to start Example 2, ask what patterns they observe in the table.</p> <ul style="list-style-type: none"> <li>• <i>The number of coasters is multiplied by 3. <math>50(3) = 150</math>, etc.</i></li> <li>• <i>Every time the cost goes up by 150, the number of coasters goes up 50. So that's like saying 50 coasters costs \$150. So each coaster cost \$3</i></li> <li>• <i>Yes, that's true in the other table. Every time the cost goes up by 60, the number of coasters goes up 20. So that's like saying 20 coasters costs \$60. So each coaster cost \$3.</i></li> </ul>	<p>Assessing Questions</p> <ul style="list-style-type: none"> <li>• <i>Tell us about the table. What does it tell you?</i></li> <li>• <i>What patterns are you observing in the cost of the plain coasters? How did you determine those values?</i></li> <li>• <i>What can you see as we move down the table that suggests the same information? Is that true in both tables? Why?</i></li> </ul> <p>Advancing Questions</p> <ul style="list-style-type: none"> <li>• <i>What equation represents the cost of any number of plain coasters?</i></li> <li>• <i>How can you use what you just determined to find the cost of coasters with a logo?</i></li> </ul>
<div style="display: flex; justify-content: space-between;"> <div style="width: 45%;"> <math display="block">y = 3x</math> <math display="block">y = 3x + 25</math> <p><math>y = 3x</math> represents a proportional relationship</p> </div> <div style="width: 50%;"> <ul style="list-style-type: none"> <li>• <i>We needed to multiply the number of coasters by 3. For the other, we then had to add 25.</i></li> <li>• <i>3 is the cost of each coaster (cost per coaster, unit rate). 25 is the extra fee or set-up fee.</i></li> <li>• <i><math>x</math> is the number of coasters and <math>y</math> is the cost of that number of coasters.</i></li> </ul> </div> </div>	<p>Example 1</p> <p>Assessing Questions</p> <ul style="list-style-type: none"> <li>• <i>Tell us about your equations. How did you determine them?</i></li> <li>• <i>Explain what the 3 and 25 refer to. What about the <math>x</math> and <math>y</math>?</i></li> <li>• <i>How do you know <math>y = 3x</math> represents a proportional relationship?</i></li> </ul> <p>Advancing Questions</p> <ul style="list-style-type: none"> <li>• <i>Where will I see the 3 on the graph of this information? The 25?</i></li> </ul>

	<ul style="list-style-type: none"><li>When you double or triple <math>x</math>, so does <math>y</math>. So (3, 9), (6, 18) and (9, 27) all work.</li></ul>	<ul style="list-style-type: none"><li>How do the graphs of each of these equations compare?</li><li>How can I tell from the graph that the relationship is proportional?</li></ul> <p>OR</p> <ul style="list-style-type: none"><li>If I build a table of values for this information, where will I see the 3? The 25?</li><li>How can I tell from the tables that the relationship is proportional?</li><li>How do the tables of each of these equations compare? What is the same? Different?</li></ul> <p>Example 1 may be solved using a table or graph. Use questions similar to the ones noted below for Example 2 as assessing and advancing questions, in that case.</p>																																																
<table><tr><td></td><td>Number of coasters</td><td>Cost of Plain Coasters</td><td></td></tr><tr><td></td><td>50</td><td>150</td><td></td></tr><tr><td>50</td><td>100</td><td>300</td><td>150</td></tr><tr><td>50</td><td>150</td><td>450</td><td>150</td></tr><tr><td>50</td><td>200</td><td>600</td><td>150</td></tr><tr><td>50</td><td>250</td><td>750</td><td>150</td></tr></table> <p><math>y = 3x</math></p> <table><tr><td></td><td>Number of coasters</td><td>Cost of Coasters With a</td><td></td></tr><tr><td></td><td>20</td><td>90</td><td></td></tr><tr><td>20</td><td>40</td><td>150</td><td>60</td></tr><tr><td>20</td><td>60</td><td>210</td><td>60</td></tr><tr><td>20</td><td>80</td><td>270</td><td>60</td></tr><tr><td>20</td><td>100</td><td>330</td><td>60</td></tr></table> <p><math>y = 3x + 30</math></p>		Number of coasters	Cost of Plain Coasters			50	150		50	100	300	150	50	150	450	150	50	200	600	150	50	250	750	150		Number of coasters	Cost of Coasters With a			20	90		20	40	150	60	20	60	210	60	20	80	270	60	20	100	330	60	<ul style="list-style-type: none"><li>We found the differences in the cost and in the number of coasters. We noticed a constant ratio, 150/50. That tells us the <math>m</math> in the equation. The data is proportional, so <math>b = 0</math>.</li><li>In the other case, the ratio was 60/20, but the data was not proportional. So we checked <math>20(3) = 60</math> and that's off by 30. So <math>b = 30</math>.</li></ul>	<p>Example 2</p> <p>Assessing Questions</p> <ul style="list-style-type: none"><li>Tell us about your strategy. Explain your thinking.</li><li>How can I tell from the tables if either relationship is proportional?</li></ul> <p>Advancing Questions</p> <ul style="list-style-type: none"><li>What do the 3 and the 30 mean in the context of the problem? Where will I see the 150 and the 50 on the graph of this information? The 3? The 30?</li><li>How do you think the graphs of each of these equations differ?</li></ul>
	Number of coasters	Cost of Plain Coasters																																																
	50	150																																																
50	100	300	150																																															
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(May be on Graphing Calculator)	<ul style="list-style-type: none"><li>We found the slope of the graph, since it went through (100,300) and (200, 600). We found the differences (counted rise and run) and formed the ratio of the change</li></ul>	<p>Assessing Questions</p> <ul style="list-style-type: none"><li>Tell about the strategy you used and about your graphs. How did they help you think about the problem?</li><li>How do you know if the data is proportional?</li></ul> <p>Advancing Questions</p>																																																



$$y = 3x$$

$$y = 3x + 30$$



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$$y = mx + b$$

$$y = 3x$$

$$y = 3x + 25$$

in  $y$  to the change in  $x$ . That was  $300/100 = 3$ . The data is proportional, so  $b = 0$

- For the second we used  $(20, 90)$  and  $(80, 270)$  in the same way. The data was not proportional. So we checked  $20(30) = 60$  and that's off by 30. So  $b = 30$ .
- We used the table to decide if the data was proportional or not. (or we extended the line, etc.)

- Can you predict what will happen to their graphs if you extend each one through its  $y$ -intercept (if no extended line)?
- What does the 3 mean in terms of the coasters? What about the 0 and the 30 (if no extended line)?
- Is it possible to graph both of these relationships on the same set of axes? Why or why not? If so, what additional information will the graph tell you?

- We used the first and last data points in the table. We subtracted the  $y$ 's, then the  $x$ 's and found the ratio of the two. That's our slope. We plugged that for  $m$  in  $y = mx + b$ . Then we checked with the first point to find  $b$ .
- We needed no correction in the first table;  $3(50) = 150$ , so the constant is 0. That says the data is proportional. In the second table, we needed to add 30, so the data is not

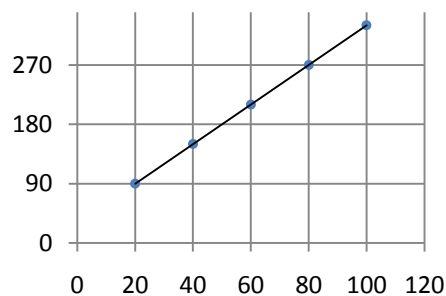
#### Assessing Questions

- Share your strategy. Explain your thinking.
- How can I tell from your equations if the relationship is proportional?

#### Advancing Questions

- Why does "needing no correction or constant" mean the data is proportional? Or needing one mean it is not proportional?
- If 3 is the slope of the line, where will I see the 3 on the graphs of these equations? What does it mean in the context of the problem? What about the 0 and the 30?
- Is it possible to graph both of these relationships on the

	<i>proportional.</i>	<i>same set of axes? Why or why not? If so, what additional information will the graph tell you?</i>
Possible Errors and Misconceptions		Possible Questions to Address Errors and Misconceptions
Reversing the “m” and the “b” in $y = mx + b$ or finding the slope by reversing the ratio to the change in x divided by the change in y.		<p>Assessing Questions</p> <ul style="list-style-type: none"> <li>• <i>Explain your strategy. What do the x and y represent in your equations?</i></li> <li>• <i>What does <math>30x</math> (or <math>1/3 x</math>) mean in the context of the problem? What will you find if you multiply 30 (or <math>1/3</math>) by x, the number of coasters?</i></li> <li>• <i>Why add 3?</i></li> </ul> <p>Advancing Questions</p> <ul style="list-style-type: none"> <li>• <i>When I look at the table, how can I predict what one coaster will cost? If I know that, how can I find the cost of 30 or 40 or x coasters?</i></li> <li>• <i>Why might I need to add (or subtract) something from mx in the equation <math>mx + b</math>? What would that mean in the context of our problem?</i></li> </ul>
SHARE DISCUSS ANALYZE PHASE: Deepening and Connecting Mathematical Understanding		
<p><b>General Considerations:</b></p> <ul style="list-style-type: none"> <li>• Orchestrate the class discussion so that it builds on, extends, and connects the thinking and reasoning of students.</li> <li>• Sequence the solution paths so that you will be able to press students to make comparisons and connections across, and between, the various solution paths.</li> </ul>		
Possible Sequence of Solution Paths	Possible Questions and Possible Student Responses	
<p>Cost in Dollars</p> <p>Number of Coasters</p>	<p>Explain your group's solution.</p> <ul style="list-style-type: none"> <li>• <i>We graphed the data and found rise/run. You can see from these two points (100, 300) and (200, 600) that it went up 300 and over 100. <math>300/100</math> is the slope. We extended the line, and it went through (0,0). So 0 is the y-intercept. So <math>y = mx + b</math> means <math>y = 3x + 0</math>.</i></li> </ul> <p>What does the 3 mean in the context of the problem?</p> <ul style="list-style-type: none"> <li>• <i>Well, since we are multiplying it by x and x is the number of coasters, it is the price of one coaster. Then we can multiply it by x coasters, and we have the cost of buying x coasters.</i></li> </ul> <p>What does the 0 mean in the context of the problem?</p> <ul style="list-style-type: none"> <li>• <i>It means 0 coasters cost 0 dollars.</i></li> </ul> <p>Tell us about your other equation.</p>	



- Same thing. We used these two points ((20, 90) and (80, 270), saw that it went up 180 and over 60.  $180/60$  is the slope. We extended the line, and it went through (0,30). So 30 is the y-intercept. So  $y = mx + b$  means  $y = 3x + 30$ . So still \$3 for each coaster, but it did not start at 0.

What does the 30 mean in the context of the problem?

- It means 0 coasters cost 30 dollars. That did not make sense until we remembered that the coasters with a logo have a set-up charge. So like in Example 1, the additional number, 30, is \$30 for setting up the logo.

Why choose to use a graph to solve this problem? How did it help you compare the cost of buying 1000 coasters at each business?

- We knew how to find slope and intercept with graphs. That gave us equations. So then we can just use the equation to find the cost of 1000 coasters.

What accounted for the difference in cost at the two businesses?

- One was \$5 more. That's the difference between \$30 and \$25, so it's the set-up fee.

		Number of coasters	Cost of Plain Coasters		
$y = 3x$		50	150		
	50	100	300	150	
	50	150	450	150	
	50	200	600	150	
$y = 3x +$	50	250	750	150	30
		Number of coasters	Cost of Coasters With a		
		20	90		
	20	40	150	60	
	20	60	210	60	
	20	80	270	60	
	20	100	330	60	

How did your group solve the problem?

- We saw that it would take too much work to extend the table to 1000, so we decided to find equations, too. Anyway, EZ Coaster was an equation. We did not want to make it into two tables and go up to 1000.

So what did you do?

- Well, the y's are going up by 150 and the x by 50, so  $150/50 = 3$ . Then we did  $y = 3x$ . Same for the other.  $60/20 = 3$ . Only this one was not proportional, so we said,  $20(3) = 60$  and that's off by 30.  $Y = 3x + 30$ .

Tell us how you decided if the data was or was not proportional.

- Well, in the first table, you can see double-double, triple-triple...

Can someone else say that in their own words? Will you point to what you mean?

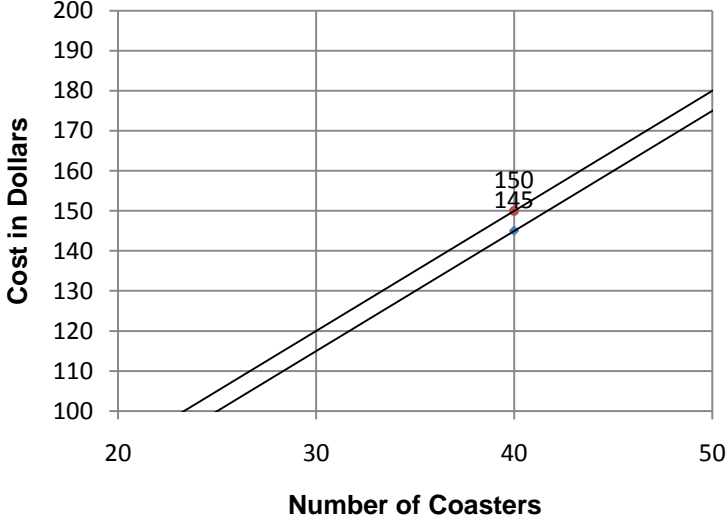
- OK, when 50 doubles to 100, 150 doubles to 300. 50 to 150 is a triple. So does 150 to 450. It's everywhere. 100 to 200 for x, but 300 to 600 for y. See? Double-double, triple-triple...

What proportion does that suggest?

- $100/50 = 300/150$ .  $150/50 = 450/150$  etc. Double, triple.

OK. How about for coasters with logo?

	<ul style="list-style-type: none"> <li>An <math>x</math> of 20 doubles to 40. But 90 does not double to 180. 20 to 60, but <math>y</math> does not triple to 270. Not proportional. <math>40/20 \neq 150/90</math></li> </ul> <p>In what ways is this tabular strategy like the graphing strategy that we just saw?</p> <ul style="list-style-type: none"> <li>In both strategies, table and graph, you can see the slope. You can get that first. Once you have that, then you can adjust for the intercept.</li> </ul> <p>Is the meaning of the slope and intercept the same in both the tabular and graphing solutions that we just saw? Can someone say what a slope of 3 means in each solution?</p> <ul style="list-style-type: none"> <li>Yes, the meaning is the same because when we relate it to the context. In both strategies, <math>3 = 3/1</math>. For each increase of 1 coaster bought the cost of the sale increases by \$3. It is a unit rate. \$3 each coaster.</li> </ul> <p>How can we tell from looking at a graph that it represents a proportional relationship?</p> <ul style="list-style-type: none"> <li>It would have to go through <math>(0,0)</math> like the first one does. If not, then you don't have the doubling and tripling that we see in the first table, because you have added something extra in only at the start of the data, not everywhere.</li> </ul>
<p>_____</p> <p>_____</p> <p><math>y = mx + b</math></p> <p><math>y = 3x</math></p> <p><math>y = 3x + 25</math></p>	<p>Just numbers here! How did your group solve the problem?</p> <ul style="list-style-type: none"> <li>We knew that we could find the slope by subtracting <math>y</math>'s, then subtracting <math>x</math>'s between the same two data points. So we did. First and last.</li> </ul> <p>Can we connect that to either the table or the graph?</p> <ul style="list-style-type: none"> <li>Yes. Subtract <math>y</math>'s, get rise. Subtract <math>x</math>'s, get run. Like on the graph.</li> </ul> <p>Will someone point to where we can see that on the graph? Yes, OK, we can see the rise and the run. How does it connect to the table??</p> <p>Well, in the first table, _____ — — All those ratios are equivalent. And the 150/50 is what that group used as the slope. The slope is proportional even in the other table. _____ — So you can use any of those ways to find the slope.</p> <p>What do you mean, the slope is proportional? How can that be, when the relationship is not?</p> <ul style="list-style-type: none"> <li>Because the slope means for each additional coaster, the price increases by \$3. So when the coasters increase to 200, the cost increases to 600. Or when the coasters go up to 20, the price goes to \$60.</li> </ul> <p>You are saying that, since the slope is a rate, there are many ways to say the</p>

	<p>same rate. We are seeing a few of them here, for our 3/1. But we do have to be careful that we go from one point to the same other point, whether we are on the graph, in the table or just subtracting. Why?</p> <ul style="list-style-type: none"> <li>Because the rate says that we compare the increase in the number of an item to the increase in its cost. So we can't use any old order, we have to keep track of how many increased, how much increased at the same time.</li> </ul>
 <p>The graph displays two linear functions representing the cost of coasters. The x-axis, labeled 'Number of Coasters', ranges from 20 to 50 with major grid lines every 10 units. The y-axis, labeled 'Cost in Dollars', ranges from 100 to 200 with major grid lines every 10 units. Two lines are plotted: a red line and a blue line. The red line passes through the points (30, 120) and (40, 150). The blue line passes through the points (30, 115) and (40, 145). The lines are parallel, indicating a constant rate of change. The red line is consistently 5 units higher than the blue line on the y-axis for any given x-value.</p>	<p>How did your group solve the problem?</p> <ul style="list-style-type: none"> <li>We decided to use a graph, but thought to put both types of coasters on the same axes. We used the graphing calculator. You can then see that one of them always costs more than the other.</li> </ul> <p>Why can you graph the coasters from both businesses on the same axes?</p> <ul style="list-style-type: none"> <li>Both are matching numbers of coasters on the x-axis to the cost on the y-axis.</li> </ul> <p>How did you use this graph to solve the problem?</p> <ul style="list-style-type: none"> <li>We had to play with the scale a bit, because at first, we thought we saw only one line. When we zoomed in, we could see both lines. So we kept going. We labeled one point so we can show which line is which. If you look closely, you can see the lines are always 5 apart.</li> </ul> <p>What does that 5 represent?</p> <ul style="list-style-type: none"> <li>One was \$5 more. That's the difference between \$30 and \$25, so it's the set-up fee. Like with the other group, that's what accounts for the difference in cost between the two businesses.</li> </ul>

CLOSURE
<p><b>Quick Write:</b></p> <ul style="list-style-type: none"> <li>Explain how to find the slope of a line using tables, graphs and numerical expressions. In what ways are these methods the same?</li> </ul>
<p><b>Possible Assessment:</b></p> <ul style="list-style-type: none"> <li>Continue to provide additional contexts where assorted representations solve the problem. Ask them to explain their thinking and choice of strategy in writing.</li> </ul>
<p><b>Homework:</b></p> <ul style="list-style-type: none"> <li>Find items from the current curriculum that will allow them to apply these ideas and understandings.</li> </ul>

## EZ Coasters

Emma and Zeke own an on-line business, EZ Coasters, that sells cork beverage coasters which can be bought plain or with logos. The price for both types of coasters is \$3.00 each. There is also a one-time set-up charge of \$25 for coasters with a logo.



1. Write an equation describing the relationship between:
  - a. the number of plain coasters bought and the cost of the coasters.
  - b. the number of coasters with a logo bought and the price of the coasters.
  - c. Is either coaster relationship proportional? Explain in writing how you know.
  - d. Explain what the slope of the line described by each equation means in the context of the coasters.
2. Coaster Center also sells coasters with and without a logo. The tables below show the cost of ordering plain coasters or coasters with a logo from their business.

**Coaster Center Price Listing**

Plain Coasters	
Number	Cost in Dollars
50	150
100	300
150	450
200	600
250	750

Coasters with a logo	
Number	Cost in Dollars
20	90
40	150
60	210
80	270
100	330

- a. Compare the cost of buying 1000 plain coasters at EZ Sales to that at Coaster Center. What accounts for any differences in price?
- b. Compare the cost of buying 1000 coasters with logo at the two businesses. What accounts for any differences in price?

3. EZ Sales also sells plain natural sandstone coasters for \$4.00 each.
- Write the equation describing this relationship, and indicate whether or not it is a proportional relationship.
  - Will the graph of this equation ever intersect the graph of the equations for either of the other two plain coasters? Explain in writing how you made your decision.
4. Using the graph below, identify the type of coaster and the company who sells it for each line graphed. Explain how you made your decisions for each line.
- Line A
  - Line B
  - Line C

